GLOBAL WELL-POSEDNESS FOR THE RADIAL DEFOCUSING CUBIC WAVE EQUATION ON $\mathbb{R}^3$ AND FOR ROUGH DATA

TRISTAN ROY

Abstract. We prove global well-posedness for the radial defocusing cubic wave equation

$$
\begin{cases}
\partial_{tt}u - \Delta u = -u^3 \\
u(0, x) = u_0(x) \\
\partial_t u(0, x) = u_1(x)
\end{cases}
$$

with data $(u_0, u_1) \in H^s \times H^{s-1}$, $1 > s > \frac{7}{10}$. The proof relies upon a Morawetz-Strauss-type inequality that allows us to control the growth of an almost conserved quantity.

1. Introduction

We shall study the defocusing cubic wave equation on $\mathbb{R}^3$

$$
\begin{cases}
\partial_{tt}u - \Delta u = -u^3 \\
u(0, x) = u_0(x) \\
\partial_t u(0, x) = u_1(x)
\end{cases}
$$

(1.1)

We shall focus on the strong solutions of the defocusing cubic wave equation on some interval $[0, T]$ i.e real-valued maps $(u, \partial_t u) \in C([0, T], H^s(\mathbb{R}^3)) \times C([0, T], H^{s-1}(\mathbb{R}^3))$ that satisfy for $t \in [0, T]$ the following integral equation

$$
u(t) = \cos(tD)u_0 + D^{-1}\sin(tD)u_1 - \int_0^t D^{-1}\sin\left((t-t')D\right)u^3(t') dt'
$$

(1.2) with $(u_0, u_1)$ lying in $H^s \times H^{s-1}$. Here $H^s$ is the usual inhomogeneous Sobolev space i.e $H^s$ is the completion of the Schwartz space $\mathcal{S}(\mathbb{R}^3)$ with respect to the norm

$$
\|f\|_{H^s} := \|(1 + D^s)f\|_{L^2(\mathbb{R}^3)}
$$

(1.3) where $D$ is the operator defined by

$$
\hat{D}f(\xi) := |\xi|\hat{f}(\xi)
$$

(1.4) and $\hat{f}$ denotes the Fourier transform

$$
\hat{f}(\xi) := \int_{\mathbb{R}^3} f(x)e^{-ix\cdot\xi} dx
$$

(1.5) Here $H^s \times H^{s-1}$ is the product space of $H^s$ and $H^{s-1}$ endowed with the standard norm $\|(f, g)\|_{H^s \times H^{s-1}} := \|f\|_{H^s} + \|g\|_{H^{s-1}}$. 

It is known [11] that (1.1) is locally well-posed in $H^s(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3)$ for $s \geq \frac{1}{2}$. Moreover if $s > \frac{1}{2}$ the time of local existence only depends on the norm of the initial data $\|(u_0, u_1)\|_{H^s \times H^{s-1}}$.

Now we turn our attention to the global well-posedness theory of (1.1). In view of the above local well-posedness theory and standard limiting arguments it suffices to establish an a priori bound of the form

$$\|u(T)\|_{H^s} + \|\partial_t u(T)\|_{H^{s-1}} \leq C(s, (\|u_0\|, \|u_1\|)_{H^s \times H^{s-1}}, T)$$

for all times $0 < T < \infty$ and all smooth-in-time Schwartz-in-space solutions $(u, \partial_t u) : [0, T] \times \mathbb{R}^3 \rightarrow \mathbb{R}$, where the right-hand side is a finite quantity depending only on $s$, $\|u_0\|_{H^s}$, $\|u_1\|_{H^{s-1}}$ and $T$. Therefore in the sequel we shall restrict attention to such smooth solutions.

The defocusing cubic wave equation (1.1) enjoys the following energy conservation law

$$E(u(t)) := \frac{1}{2} \int_{\mathbb{R}^3} (\partial_t u)^2(x, t) \, dx + \frac{1}{4} \int_{\mathbb{R}^3} |Du(x, t)|^2 \, dx + \frac{1}{4} \int_{\mathbb{R}^3} u^4(x, t) \, dx$$

Combining this conservation law to the local well-posedness theory we immediately have global well-posedness for (1.1) and for $s = 1$.

In this paper we are interested in studying global well-posedness for (1.1) and for data below the energy norm, i.e $s < 1$. It is conjectured that (1.1) is globally well-posed in $H^s(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3)$ for all $s > \frac{1}{2}$. The global existence for the defocusing cubic wave equation has been the subject of several papers. Let us some mention some results for data lying in a slightly different space than $H^s \times H^{s-1}$, i.e $\dot{H}^s \times \dot{H}^{s-1}$. Here $\dot{H}^s$ is the usual homogeneous Sobolev space i.e the completion of Schwartz functions $S(\mathbb{R}^3)$ with respect to the norm

$$\|f\|_{\dot{H}^s} = \|D^s f\|_{L^2(\mathbb{R}^3)}$$

Kenig, Ponce and Vega [9] were the first to prove that (1.1) is globally well-posed for $1 > s > \frac{3}{4}$. They used the Fourier truncation method discovered by Bourgain [2]. I. Gallagher and F. Planchon [7] proposed a different method to prove global well-posedness for $1 > s > \frac{3}{4}$. H. Bahouri and Jean-Yves Chemin [1] proved global-wellposedness for (1.1) and for $s = \frac{3}{4}$ by using a non linear interpolation method and logarithmic estimates from S. Klainermann and D. Tataru [10]. We shall consider global well-posedness for the radial defocusing cubic wave equation i.e global existence for the initial value problem (1.1) with radial data. The main result of this paper is the following one

**Theorem 1.1.** The radial defocusing cubic wave equation is globally well-posed in $H^s \times H^{s-1}$ for $1 > s > \frac{1}{10}$. Moreover if $T$ large then

$$\|u(T)\|_{H^s}^2 + \|\partial_t u(T)\|_{H^{s-1}}^2 \leq C(\|u_0\|_{H^s}, \|u_1\|_{H^{s-1}}) T^{\frac{16}{3} + \frac{10}{9}}$$

for $\frac{5}{6} \geq s > \frac{7}{10}$ and

$$\|u(T)\|_{H^s}^2 + \|\partial_t u(T)\|_{H^{s-1}}^2 \leq C(\|u_0\|_{H^s}, \|u_1\|_{H^{s-1}}) T^{\frac{2s}{3} + 1}$$
for $1 > s > \frac{5}{6}$. Here $C (\| u_0 \|_{H^s}, \| u_1 \|_{H^{s-1}})$ is a constant only depending on $\| u_0 \|_{H^s}$ and $\| u_1 \|_{H^{s-1}}$.

We set some notation that appear throughout the paper. Given $A, B$ positive number $A \lesssim B$ means that there exists a universal constant $K$ such that $A \leq KB$. We say that $K_0$ is the constant determined by the relation $A \lesssim B$ if $K_0$ is the smallest $K$ such that $A \leq KB$ is true. We write $A \sim B$ when $A \lesssim B$ and $B \lesssim A$. $A \ll B$ denotes $A \leq KB$ for some universal constant $K < \frac{1}{100}$. We also use the notations $A+ = A + \epsilon$, $A- = A - \epsilon$ for some universal constant $0 < \epsilon << 1$. Let $\nabla$ denote the gradient operator. If $J$ is an interval then $|J|$ is its size. If $E$ is a set then $\text{card}(E)$ is its cardinal. Let $I$ be the following multiplier

\begin{equation}
(1.11) \quad \hat{f}(\xi) := m(\xi) \hat{f}(\xi)
\end{equation}

where $m(\xi) := \eta \left( \frac{\xi}{N} \right)$, $\eta$ is a smooth, radial, nonincreasing in $|\xi|$ such that

\begin{equation}
(1.12) \quad \eta(\xi) := \begin{cases} 
1, & |\xi| \leq 1 \\
\left( \frac{1}{|\xi|} \right)^{1-s}, & |\xi| \geq 2 
\end{cases}
\end{equation}

and $N >> 1$ is a dyadic number playing the role of a parameter to be chosen. We shall abuse the notation and write $m(|\xi|)$ for $m(\xi)$, thus for instance $m(N) = 1$.

We recall some basic results regarding the defocusing cubic wave equation. Let $\lambda \in \mathbb{R}$ and $u_\lambda$ denote the following function

\begin{equation}
(1.13) \quad u_\lambda(t, x) := \frac{1}{\lambda} u \left( \frac{t}{\lambda}, \frac{x}{\lambda} \right)
\end{equation}

If $u$ satisfies (1.1) with data $(u_0, u_1)$ then $u_\lambda$ also satisfies (1.1) but with data $\left( \frac{1}{\lambda} u_0 \left( \frac{x}{\lambda} \right), \frac{1}{\lambda} u_1 \left( \frac{x}{\lambda} \right) \right)$. If $u$ satisfies the radial defocusing cubic wave equation then $u$ is radial.

Now we recall some standard estimates that we use later in this paper.

**Proposition 1.2. "Strichartz estimates in 3 dimensions" (See [8], [11]).** Let $m \in [0, 1]$. If $u$ is a strong solution to the IVP problem

\begin{equation}
(1.14) \quad \begin{cases} 
\partial_t u - \Delta u = F \\
u(0, x) = f(x) \in \dot{H}^m \\
\partial_t u(0, x) = g(x) \in \dot{H}^{m-1}
\end{cases}
\end{equation}

then we have for $0 \leq \tau < \infty$

\begin{equation}
(1.15) \quad \| u \|_{L_t^q L_x^r([0, \tau])} + \| u \|_{C([0, \tau]; H^m)} + \| \partial_t u \|_{C([0, \tau]; H^{m-1})} \lesssim \| f \|_{\dot{H}^m} + \| g \|_{\dot{H}^{m-1}} + \| F \|_{L_t^q L_x^r([0, \tau])}
\end{equation}

under two assumptions

- $(q, r)$ lie in the set $\mathcal{W}$ of wave-admissible points i.e

\begin{equation}
(1.16) \quad \mathcal{W} := \left\{ (q, r) \in (2, \infty] \times [2, \infty), \frac{1}{q} + \frac{1}{r} \leq \frac{1}{2} \right\}
\end{equation}
• \((\tilde{q}, \tilde{r})\) lie in the dual set \(W'\) of \(W\) i.e
\[
W' := \left\{ (\tilde{q}, \tilde{r}) : \frac{1}{q} + \frac{1}{\tilde{r}} = 1, \frac{1}{r} + \frac{1}{\tilde{r}} = 1, (q, r) \in W \right\}
\]

• \((q, r, \tilde{q}, \tilde{r})\) satisfy the dimensional analysis conditions
\[
\frac{1}{q} + \frac{3}{r} = \frac{3}{2} - m
\]

and
\[
\frac{1}{\tilde{q}} + \frac{3}{\tilde{r}} - 2 = \frac{3}{2} - s
\]

We also have the well-known estimate

**Proposition 1.3. "Radial Sobolev inequality"** If \(u : \mathbb{R}^3 \to \mathbb{C}\) is radial and smooth
\[
|u(x)| \lesssim \frac{\|u\|_{H^1}}{|x|^s}
\]

The Hardy-type inequality is proved in [3]

**Proposition 1.4. "Hardy-type inequality"** If \(1 < p < 3\) and \(u : \mathbb{R}^3 \to \mathbb{C}\) is smooth
\[
\|\frac{u}{|x|^m}\|_{L^p} \leq \frac{3}{2-p}\|Df\|_{L^p}
\]

Some variables appear frequently in this paper. We define them now.

We say that \((q, r)\) is a \(m\)-wave admissible pair if \(0 \leq m \leq 1\) and \((q, r)\) satisfy the two following conditions

• \((q, r) \in W\)
  • \(\frac{1}{q} + \frac{3}{r} = \frac{3}{2} - m\)

Let \(J = [a, b]\) be an interval included in \([0, \infty)\). Let \(Z_{m,s}(J)\) denote the following number
\[
Z_{m,s}(J) := \sup_{q,r} (\|D^{1-m}Iu\|_{L^q(J)L^r_x} + \|D^{-m}I\partial_t u\|_{L^q(J)L^r_x})
\]
where the sup is taken over \(m\)-wave admissible \((q, r)\) and let
\[
Z(J) := \sup_{m \in [0,1]} Z_{m,s}(J)
\]

Let
\[
R_1(J) := \int_J \int_{\mathbb{R}^3} \frac{\nabla Iu(t,x) \cdot x}{|x|} ((Iu)^3(t,x) - Iu^3(t,x)) \, dx \, dt
\]
and
\[
R_2(J) := \int_J \int_{\mathbb{R}^3} \frac{Iu(t,x)}{|x|} ((Iu)^3(t,x) - Iu^3(t,x)) \, dx \, dt
\]
If \(J = [0, \tau]\) we shall abuse the notation and write
that we can control the number of intervals \( J \) constraints as \( J \)

\[ Z(\tau) := Z(J) \quad R(\tau) := R(J) \]

Some estimates that we establish throughout the paper require a Paley-Littlewood
decomposition. We set it up now. Let \( \phi(\xi) \) be a real, radial, nonincreasing function
that is equal to 1 on the unit ball \( \{ \xi \in \mathbb{R}^3 : |\xi| \leq 1 \} \) and that is supported on
\( \{ \xi \in \mathbb{R}^3 : |\xi| \leq 2 \} \). Let \( \psi \) denote the function

\[ \psi(\xi) := \phi(\xi) - \phi(2\xi) \]

If \( M \in 2^\mathbb{Z} \) is a dyadic number we define the Paley-Littlewood operators in the
Fourier domain by

\[
\begin{align*}
P_{\leq M} f(\xi) & := \phi \left( \frac{\xi}{M} \right) \hat{f}(\xi) \\
P_M f(\xi) & := \psi \left( \frac{\xi}{M} \right) \hat{f}(\xi) \\
P_{> M} f(\xi) & := \hat{f}(\xi) - P_M f(\xi)
\end{align*}
\]

Since \( \sum_{M \in 2^\mathbb{Z}} \psi \left( \frac{\xi}{M} \right) = 1 \) we have

\[ f = \sum_{M \in 2^\mathbb{Z}} P_M f \]

We conclude this introduction by giving the main ideas of the proof of theorem 1.1 and explaining how the paper is organized. Following the proof of the
global well-posedness for \( s = 1 \) we try to compare for every \( T > 0 \) the relevant
quantity \( \| (u(T), \partial_t u(T)) \|_{H^s \times H^{s-1}} \) to the supremum of the energy conservation
law \( \sup_{t \in [0,T]} E(u(t)) \). Unfortunately this strategy does not work if \( s < 1 \) since the
energy can be infinite. We get around this difficulty by using the I-method designed
by J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao [5] and successfully
applied to prove global well-posedness for semilinear Schrödinger equations and for
rough data. The idea consists of introducing the following smoothed energy

\[ E(Iu(t)) := \frac{1}{2} \int_{\mathbb{R}^3} |\partial_t Iu(x,t)|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} |DIu(x,t)|^2 dx + \frac{1}{4} \int_{\mathbb{R}^3} |Iu(x,t)|^4 dx \]

We prove in section 3 that \( \| (u(T), \partial_t u(T)) \|_{H^s \times H^{s-1}}^2 \) and the supremum of the
smoothed energy on \( [0, T] \) are comparable. Therefore we try to estimate \( \sup_{t \in [0,T]} E(Iu(t)) \)
in order to give an upper bound of \( \|(u(T), \partial_t u(T))\|_{H^s \times H^{s-1}} \). For convenience we
place the mollified energy at time zero into \( [0, \frac{1}{T}] \) by choosing the right scaling
factor \( \lambda \). This operation shows that we are reduced to estimate \( \sup_{t \in [0, \lambda T]} E(Iu_\lambda(t)) \).
In section 4 we prove that we can locally control a variable namely \( Z(J) \) provided
that the interval \( J \) satisfies some constraints that give some information about its
size. \( \sup_{t \in J} E(Iu_\lambda(t)) \) is estimated by the fundamental theorem of calculus. The
upper bound depends on the parameter \( N \) and the controlled quantity \( Z(J) \). This
estimate is established in section 5. Now we can iterate: the process generates a
sequence of intervals \( (J_i) \) that cover the whole interval \( [0, \lambda T] \) and satisfy the same
constraints as \( J \). We should be able to estimate \( \sup_{t \in [0, \lambda T]} E(Iu_\lambda(t)) \) provided
that we can control the number of intervals \( J_i \). This requires the establishment
of a long time estimate, the so-called almost Morawetz-Strauss inequality. This
estimate is proved in section 6. It depends on some remainder integrals that are estimated in section 7. Combining this inequality to the radial Sobolev inequality (1.20) we can give an upper bound of the cardinal of \((J_i)\). The proof of theorem 1.1 is given in section 2.

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2. Proof of global well-posedness for \(1 > s > \frac{7}{10}\)

In this section we prove the global existence of (1.1) for \(1 > s > \frac{7}{10}\). Our proof relies on some intermediate results that we prove in later sections. More precisely we shall show the following

**Proposition 2.1. ”\(H^s\) norms and mollified energy estimates”** Let \(T > 0\).

\[
\|u(T)\|_{H^s}^2 + \|\partial_t u(T)\|_{H^{s-1}}^2 \lesssim \|u_0\|_{H^s}^2 + (T^2 + 1) \sup_{t \in [0,T]} E(Iu(t))
\]

for every \(u\).

**Proposition 2.2. ”Local boundedness”** Let \(J = [a,b]\) be an interval included in \([0, \infty)\). Assume that \(E(Iu(a)) \leq 2\) and that \(u\) satisfies (1.1). There exist \(C_1, C_2\) small and positive constants such that if \(J\) satisfies

\[
\|Iu\|_{L^6_x(J)L^6_t} \leq \frac{C_1}{|J|^{\frac{3}{2}}}
\]

and

\[
|J| \leq C_2 N^{\frac{s}{s-\frac{5}{2}}}
\]

then we have

\[
Z(J) \lesssim 1
\]

**Proposition 2.3. ”Almost conservation law”** Let \(J = [a,b]\) be an interval included in \([0, \infty)\). Assume that \(u\) satisfies (1.1). Then we have

\[
\left| \sup_{t \in J} E(Iu(t)) - E(Iu(a)) \right| \lesssim \frac{Z^4(J)}{N^5}
\]

**Proposition 2.4. ”Almost Morawetz-Strauss inequality”** Let \(T \geq 0\). Assume that \(u\) satisfies (1.1). Then we have

\[
\int_0^T \int_{\mathbb{R}^3} \frac{|Iu|^4(t,x)}{|x|} dx dt - 2 \left( E(Iu(0)) + E(Iu(T)) \right) \lesssim |R_1(T)| + |R_2(T)|
\]

and
Proposition 2.5. "Estimate of integrals" Let \( J \) be an interval included in \([0, \infty)\). Then if \( i = 1, 2 \) we have

\[
R_i(J) \lesssim \frac{Z^4(J)}{N^{1/2}}.
\]

For the remainder of the section we show how proposition 2.2, 2.3, 2.4 and 2.5 imply Theorem 1.1.

Let \( T > 0 \) and \( N = N(T) >> 1 \) be a parameter to be chosen later. There are three steps to prove Theorem 1.1.

1. **Scaling.** Let \( \lambda >> 1 \) to be chosen later. Then by Plancherel theorem

\[
\| DIu_\lambda(0) \|_{L^2}^2 \lesssim \int_{|\xi| \leq 2N} |\xi|^2 |\hat{u}_\lambda(0, \xi)|^2 \, d\xi + \int_{|\xi| \geq 2N} \frac{N^{2(1-s)}}{|\xi|^{2(1-s)}} |\hat{u}_\lambda(0, \xi)|^2 \, d\xi
\]

\[
\lesssim N^{2(1-s)} \| u_\lambda(0) \|_{H^s}^2
\]

\[
\lesssim N^{2(1-s)} \lambda^{1-2s} \| u_0 \|_{H^s}^2
\]

By homogeneous Sobolev embedding

\[
\| Iu_\lambda(0) \|_{L^4}^4 \lesssim \int_{|\xi| \leq 2N} |\xi|^{\frac{4}{3}} |\hat{Iu}_\lambda(0, \xi)|^2 \, d\xi
\]

\[
\lesssim \int_{|\xi| \leq 2N} \frac{N^{2(1-s)}}{|\xi|^{2(1-s)}} |\hat{Iu}_\lambda(0, \xi)|^2 \, d\xi
\]

\[
\lesssim \max \left( \frac{N^{2(1-s)}}{\lambda^2}, \frac{N^{2(1-s)}}{\lambda^{2(1-s)}} \right) \| u_0 \|_{H^s}^2 + N^{\frac{4}{5} - 2s} \lambda^{1-2s} \| u_0 \|_{H^s}^2.
\]

Hence

\[
\| Iu_\lambda(0) \|_{L^4}^4 \lesssim N^{2(1-s)} \lambda^{1-2s} \| u_0 \|_{H^s}^4.
\]

By (2.8), (2.9) and (2.11) we see that there exists \( C_0 = C_0 (\| u_0 \|_{H^s}, \| u_1 \|_{H^{s-1}}) \) such that if \( \lambda \) satisfies

\[
\lambda = C_0 N^{\frac{2(1-s)}{1-s}}
\]

then

\[
E(Iu_\lambda(0)) \leq \frac{1}{\ell^2}
\]
(2) **Boundedness of the mollified energy.** Let $F_T$ denote the following set

$$
F_T = \left\{ T' \in [0, T] : \sup_{t \in [0, \lambda T']} E(Iu_\lambda(t)) \leq 1 \text{ and } \|Iu_\lambda\|_{L^6_\ast([0, \lambda T']) L^6_\ast} \leq (16C^2_\ast)^+ + 1 \right\}
$$

with $C_\ast$ being the constant determined by $\lesssim$ in (1.20) and $\lambda$ satisfying (2.12). We claim that $F_T$ is the whole set $[0, T]$ for $N = N(T) \gg 1$ to be chosen later. Indeed

- $F_T \neq \emptyset$ since $0 \in F_T$ by (2.13).
- $F_T$ is closed by continuity and by the dominated convergence theorem
- $F_T$ is open. Let $T' \in F_T$. By continuity there exists $\delta > 0$ such that for every $T' \in (T' - \delta, T' + \delta) \cap [0, T]$ we have

$$
\sup_{t \in [0, \lambda T']} E(Iu_\lambda(t)) \leq 2
$$

and

$$
\|Iu_\lambda\|_{L^6_\ast([0, \lambda T']) L^6_\ast} \leq (16C^2_\ast)^+ + 2
$$

We are interested in generating a partition $\{J_j\}$ of $[0, \lambda T']$ such that (2.2) and (2.3) are satisfied for all $J_j$. We describe now the algorithm.

**Description of the algorithm.** Let $\mathcal{L}$ be the present list of intervals. Let $L$ be the sum of the lengths of the intervals making up $\mathcal{L}$. Let $n$ be the number of the last interval of $\mathcal{L}$. Initially there is no interval and we start from the time $t = 0$. Therefore $\mathcal{L}$ is empty and we assign the value 0 to $L$ and $n$. Then as long as $L < \lambda T'$ do the following

(a) consider $f_L(\tau) = \|Iu_\lambda\|_{L^6_\ast([L, L+\tau]) L^6_\ast} - \frac{C_1}{\tau^\frac{1-\epsilon}{2}}, \tau \geq 0$ with $C_1$ defined in (2.2).

(b) since $f_L$ is continuous, does not decrease and $f_L(\tau) \to -\infty$ as $\tau \to 0$, $\tau \geq 0$ there are two options

- $f_L$ is always negative on $[0, \lambda T' - L]$; in this case if (2.3) is satisfied by $[L, \lambda T']$ then let $J_n := [L, \lambda T']$. If not let $J_n := [L, L + C_2 N^{\frac{1+\epsilon}{2}}]$.

- $f_L$ has one and only one root on $[0, \lambda T' - L]$; in this case let $\tau_0$ be this root. If (2.3) is satisfied by $[L, L + \tau_0]$ then let $J_n := [L, L + \tau_0]$. If not let $J_n := [L, L + C_2 N^{\frac{1-\epsilon}{2}}]$.

(c) assign the value $L + |J_n|$ to $L$.

(d) assign the value $n + 1$ to the variable $n$

(e) insert $J_n$ into $\mathcal{L}$ so that $\mathcal{L} = (J_j)_{j \in \{1, \ldots, n\}}$

When we apply this algorithm it is not difficult to see that

- $\|Iu_\lambda\|_{L^6_\ast(J_j) L^6_\ast} = \frac{C_1}{|J_j|^\frac{1-\epsilon}{2}}$ or $|J_j| = C_2 N^{\frac{1+\epsilon}{2}}$ for every $j \in \{1, \ldots, \text{card}(\mathcal{L}) - 1\}$
- $J_j \cap J_k = \emptyset$ for every $(j, k) \in \{1, \ldots, \text{card}(\mathcal{L})\}^2$ such that $j \neq k$
- $\bigcup_{j=1}^{\text{card}(\mathcal{L})} J_j$ is a left-closed interval with left endpoint 0 and included in $[0, \lambda T']$. Moreover $\bigcup_{j=1}^{\text{card}(\mathcal{L})} J_j = [0, \lambda T']$ if the process is finite.
Let

\[(2.17)\quad \mathcal{L}_1 = \left\{ J_j, J_j \in \mathcal{L}, ||Iu||^6_{L^6([0, \lambda T])} = \frac{C_1}{|J_j|^3} \right\}\]

and

\[(2.18)\quad \mathcal{L}_2 = \left\{ J_j, J_j \in \mathcal{L}, |J_j| = C_2 N^{\frac{1-s}{2}} \right\}\]

We have \((J_j)_{j=1}^{\text{card}(\mathcal{L})} \subset \mathcal{L}_1 \cup \mathcal{L}_2\). We claim that \(\text{card}(\mathcal{L}) < \infty\), \(i = 1, 2\). If not let us consider the \(m_1, m_2\) first elements of \(\mathcal{L}_1, \mathcal{L}_2\) respectively. Then

\[(2.19)\quad m_1 C_2 N^{\frac{1-s}{2}} \leq \lambda T'\]

By Hölder inequality and by (2.16) we have

\[(2.20)\quad m_2 = \sum_{j=1}^{m_2} |J_j|^{-\frac{2}{3}}|J_j|^\frac{2}{3} \leq \left( \sum_{j=1}^{m_2} \frac{1}{|J_j|} \right)^{\frac{1}{2}} \left( \sum_{j=1}^{m_2} |J_j| \right)^{\frac{1}{2}} \leq ||Iu||^2_{L^6([0, \lambda T])} \left( \lambda T' \right)^{\frac{2}{3}} \lesssim (\lambda T')^{\frac{2}{3}}\]

Letting \(m_1\) and \(m_2\) go to infinity in (2.19) and (2.20) we have a contradiction. Therefore \(\text{card}(\mathcal{L}) < \infty\) and \(\bigcup_{j=1}^{\text{card}(\mathcal{L})} J_j = [0, \lambda T']\). Moreover we have by (2.12), (2.19) and (2.20)

\[(2.21)\quad \text{card}(\mathcal{L}) \lesssim (\lambda T')^{\frac{2}{3}} + \frac{\lambda T'}{N^{\frac{s-1}{2}}} + 1 \lesssim N^{\frac{4(1-s)}{3} T^\frac{2}{3} + T + 1}\]

Now by (2.13), (2.15), (2.21), proposition 2.2, 2.3, 2.4 and 2.5 we get after iterating

\[(2.22)\quad \sup_{t \in [0, \lambda T']} E(Iu \lambda(t)) - \frac{1}{2} \lesssim \frac{N^{\frac{4(1-s)}{3} T^\frac{2}{3} + T + 1}}{N^s + T^2 + 1}\]

and

\[(2.23)\quad \int_0^{\lambda T'} \int_{\mathbb{R}^3} \frac{|Iu \lambda(t,x)|^4}{|x|} \, dx \, dt - 2 \left( E(Iu \lambda(\lambda T')) + E(Iu \lambda(0)) \right) \lesssim \sum_{i=1}^{\text{card}(\mathcal{L}_i)} R_i(J_j) \lesssim \frac{N^{\frac{4(1-s)}{3} T^\frac{2}{3} + T + 1}}{N^s + T^2 + 1}\]

By (1.20), (2.15), (2.23) and the inequality \((1 + x)\frac{2}{3} \leq 1 + x, x \geq 0\)

\[(2.24)\quad ||Iu\lambda||^6_{L^6([0, \lambda T']) L^6_x} = (16C_s^2)^{\frac{1}{3}} \lesssim \frac{N^{\frac{4(1-s)}{3} T^\frac{2}{3} + T + 1}}{N^s + T^2 + 1}\]
Let $C', C''$ be the constant determined by $\lesssim$ in (2.22), (2.24) respectively. Since $s > \frac{7}{10}$ we can always choose for every $T > 0$ a $N = N(T) >> 1$ such that

\begin{align}(2.25)\quad \max \left( C', C'' \right)_{N^{-\frac{4(1-s)}{2}}} T^{2} \leq \frac{1}{6} \end{align}

\begin{align}(2.26)\quad \max \left( C', C'' \right)_{N^{-1}} T \leq \frac{1}{6} \end{align}

and

\begin{align}(2.27)\quad \max \left( C', C'' \right)_{N^{-1}} \leq \frac{1}{6} \end{align}

By (2.22), (2.24), (2.25), (2.26) and (2.27) we have $\sup_{t \in [0, T]} E(Iu, (t)) \leq 1$ and $\| u \| _{L^{2}} E(Iu, (0, x)) \leq (16C_{s}^{2})^{\frac{1}{3}} + 1$.

Hence $F_{T} = [0, T]$ with $N = N(T)$ satisfying (2.25), (2.26) and (2.27).

3. Conclusion. Following the $I$- method described in [5]

\begin{align}(2.28)\quad \sup_{t \in [0, T]} E(Iu, (t)) = \lambda \sup_{t \in [0, T]} E(Iu, (t)) \lesssim \lambda \sup_{t \in [0, \lambda T]} E(Iu, (t)) \lesssim \lambda \end{align}

Combining (2.28) and proposition 2.1 we have global well-posedness.

Now let $T$ be large. If $\frac{5}{6} \geq s > \frac{7}{10}$ then let $N$ such that

\begin{align}(2.29)\quad \frac{0.9}{6} \leq \max \left( C', C'' \right)_{1} \leq \frac{1}{6} \end{align}

Notice that (2.26) and (2.27) are also satisfied. We plug (2.29) into (2.28) and we apply proposition 2.1 to get (1.9). If $1 > s > \frac{3}{6}$ then let $N$ such that

\begin{align}(2.30)\quad \frac{0.9}{6} \leq \max \left( C', C'' \right)_{T} \leq \frac{1}{6} \end{align}

Notice that (2.25) and (2.27) are also satisfied. We plug (2.30) into (2.28) and we apply proposition 2.1 to get (1.10).

3. Proof of the $H^{s}$ norms and mollified energy estimates

In this section we are interested in proving proposition 2.1. By Plancherel theorem

\begin{align} \| u(T) \| _{H^{s}}^{2} \lesssim \| P_{\leq 1} u(T) \| _{H^{s}}^{2} + \int\int_{1 \leq |\xi| \leq 2N} |\xi|^{2s} |\hat{u}(T, \xi)|^{2} d\xi + \int_{|\xi| \geq 2N} |\xi|^{2s} |\hat{u}(T, \xi)|^{2} d\xi \end{align}

But

\begin{align}(3.1)\quad \int_{1 \leq |\xi| \leq 2N} |\xi|^{2s} |\hat{u}(T, \xi)|^{2} d\xi \leq \int_{|\xi| \leq 2N} |\xi|^{2} |\hat{u}(T, \xi)|^{2} d\xi \lesssim \int_{|\xi| \geq 2N} |\hat{u}(T, \xi)|^{2} d\xi \lesssim \int_{|x| \geq 3} |D^{2} u(T, x)|^{2} dx \lesssim E(Iu(T)) \end{align}
\[
\int_{|\xi| \geq 2N} |\xi|^{2s} |\hat{u}(T, \xi)|^2 d\xi \leq \int_{|\xi| \geq 2N} |\xi|^{2(N^{2(1-s)}_2)} |\hat{u}(T, \xi)|^2 d\xi \\
\lesssim \int_\mathbb{R}^3 |D I u(T, x)|^2 dx \\
\lesssim E(I u(T))
\]  

and by the fundamental theorem of calculus and Minkowski inequality

\[
\|P_{\leq 1} u(T)\|_{H^s} \lesssim \|P_{\leq 1} u_0\|_{H^s} + \int_0^T \|P_{\leq 1} \partial_t u(t)\|_{H^s} dt \\
\lesssim \|u_0\|_{H^s} + T \sup_{t \in [0, T]} \|\partial_t I u(t)\|_{L^2}
\]

which implies that

\[
\|P_{\leq 1} u(T)\|_{H^s}^2 \lesssim \|u_0\|_{H^s}^2 + T^2 \sup_{t \in [0, T]} E(I u(t))
\]

We also have

\[
\|\partial_t u(T)\|_{H^{s-1}}^2 \lesssim E(I u(T))
\]

Combining (3.1), (3.2), (3.4) and (3.5) we get (2.1).

4. Proof of the local boundedness estimate

We are interested in proving proposition 2.2 in this section. In what follows we also assume that \( J = [0, \tau] \): the reader can check after reading the proof that the other cases can be reduced to that one.

Before starting the proof let us state the following lemma

**Lemma 4.1. "Strichartz estimates with derivative"** Let \( m \in [0, 1] \) and \( 0 \leq \tau < \infty \). If \( u \) satisfies the IVP problem

\[
\begin{align*}
\square u &= F \\
u(t = 0) &= f \\
\partial_t u(t = 0) &= g
\end{align*}
\]

then we have the \( m \)-Strichartz estimate with derivative

\[
\|u\|_{L^q(\mathbb{R}; L^r)} + \|\partial_t D^{-1} u\|_{L^q(\mathbb{R}; L^r)} \lesssim \|f\|_{H^m} + \|g\|_{H^{m-1}} + \|F\|_{L^q(\mathbb{R}; L^r)}
\]

for \( (q, r) \in \mathcal{W}, (\tilde{q}, \tilde{r}) \in \tilde{\mathcal{W}} \) and \( (q, r, \tilde{q}, \tilde{r}) \) satisfying the gap condition

\[
\frac{1}{q} + \frac{3}{r} = \frac{3}{2} - m = \frac{1}{\tilde{q}} + \frac{3}{\tilde{r}} - 2
\]

We postpone the proof of lemma 4.1 to subsection 4.1. Assuming that is true we now show how lemma 4.1 implies proposition 2.2.

Multiplying the \( m \)-Strichartz estimate with derivative (4.1) by \( D^{1-m} I \) we get

\[
Z_{m,s}(\tau) \lesssim \|D I u_0\|_{L^2} + \|I u_1\|_{L^2} + \|D^{1-m} I F\|_{L^q_t L^r_x}
\]

\[
\lesssim 1 + \|D^{1-m} I F\|_{L^q_t L^r_x}
\]

The remainder of proof is divided into three steps.
• First Step First we assume that $m \leq s$. Notice that the point $(\frac{1}{r-s}, 6)$ is $s$-wave admissible. In this case we get from the fractional Leibnitz rule the H"older in time and the H"older in space inequalities

\begin{equation}
(4.4) \quad Z_{m,s}(\tau) \lesssim 1 + \|D^{1-m}I(uuu)\|_{L^1([-\infty, 0] \cap L^\frac{6}{5-s})} \\
\lesssim 1 + \|D^{1-m}u\|_{L^\infty([-\infty, 0]) L^\frac{5}{2m}} \|u\|^2_{L^2([-\infty, 0]) L^s} \\
\lesssim 1 + Z_{m,s}(\tau) \left( \tau^{\frac{1}{s}} \|P\lesssim u\|_{L^s([-\infty, 0]) L^s} + \tau^{s-\frac{1}{2}} \|P\gtrsim u\|_{L^s([-\infty, 0]) L^s} \right) \\
\lesssim 1 + Z_{m,s}(\tau) \left( \tau^{\frac{1}{s}} \|u\|_{L^s([-\infty, 0]) L^s} + \tau^{s-\frac{1}{2}} Z_{m,s}(\tau) \right)^2 \\
\lesssim 1 + Z_{m,s}(\tau) \left( \tau^{\frac{1}{s}} \|u\|_{L^s([-\infty, 0]) L^s} + \tau^{s-\frac{1}{2}} Z_{m,s}(\tau) \right)^2
\end{equation}

Assume $m = s$. Then if we apply a continuity argument to (4.4) we get from the inequalities (2.2) and (2.3)

\begin{equation}
(4.5) \quad Z_{s,s}(\tau) \lesssim 1
\end{equation}

Now assume $m < s$. Then if we apply a continuity argument to (4.4) and the inequalities (2.2) and (4.5) we get

\begin{equation}
(4.6) \quad Z_{m,s}(\tau) \lesssim 1
\end{equation}

• Second Step We assume $m > s$. By (4.4), (4.5), (4.6), (2.2) and (2.3) we have

\begin{equation}
(4.7) \quad \|D^{1-r}I(uuu)\|_{L^1([-\infty, 0]) L^\frac{6}{5-r}} \lesssim Z_{r,s}(\tau) \left( \tau^{\frac{1}{s}} \|u\|_{L^s([-\infty, 0]) L^s} + \tau^{s-\frac{1}{2}} Z_{m,s}(\tau) \right)^2 \\
\lesssim 1
\end{equation}

for $r \leq s$. The inequality

\begin{equation}
(4.8) \quad \|D^{1-m}I(uuu)\|_{L^1([-\infty, 0]) L^\frac{6}{5-m}} \lesssim \|D^{1-r}I(uuu)\|_{L^1([-\infty, 0]) L^\frac{6}{5-r}}
\end{equation}

follows from the application of Sobolev homogeneous embedding. We get from (4.3), (4.7) and (4.8)

\begin{equation}
(4.9) \quad Z_{m,s}(\tau) \lesssim 1 + \|D^{1-m}I(uuu)\|_{L^1([-\infty, 0]) L^\frac{6}{5-m}} \\
\lesssim 1 + \|D^{1-r}I(uuu)\|_{L^1([-\infty, 0]) L^\frac{6}{5-r}} \\
\lesssim 1
\end{equation}

4.1. Proof of Lemma 4.1. We prove lemma 4.1 in this subsection.

By decomposition it suffices to prove that $u_1(t) = e^{\pm iD} f$, $u_2(t) = e^{\pm itD} g$ and $u_n(t) = \int_0^t D^{-1} \sin \left( (t-t')D \right) F dt'$ satisfy (4.1).
We have $\partial_t u_1^1(t) = \pm iDe^{\pm itD}f$ and $\partial_t u_1^2 = \pm e^{\pm itD}g$. We know from the Strichartz estimates that

$$\|D^{-1} \partial_t u_1^1\|_{L^q_t([0, \tau]) L^2_x} \lesssim \|e^{\pm itD} f\|_{L^q_t([0, \tau]) L^2_x} \lesssim \|f\|_{\dot{H}^{1/2}}$$

(4.10)

and

$$\|D^{-1} \partial_t u_1^2\|_{L^q_t([0, \tau]) L^2_x} = \|e^{\pm itD} D^{-1} g\|_{L^q_t([0, \tau]) L^2_x} \lesssim \|D^{-1} g\|_{\dot{H}^{1/2}} \lesssim \|g\|_{\dot{H}^{1/2}}$$

(4.11)

We also have

$$D^{-1} \partial_t u_n(t) = \int_0^t \cos \left( (t-t') D \right) F(t') \, dt'$$

(4.12)

and by the Strichartz estimates

$$\|D^{-1} \partial_t u_n\|_{L^q_t([0, \tau]) L^2_x} \lesssim \| \int_0^t D^{-1} e^{i(t-t') D} F(t') \, dt\|_{L^q_t([0, \tau]) L^2_x} \lesssim \|F\|_{L^q_t([0, \tau]) L^2_x}$$

(4.13)

(4.1) follows from (4.10), (4.11) and (4.13).

5. Proof of almost conservation law

Now we prove proposition 2.3. In what follows we also assume that $J = [0, \tau]$: the reader can check after reading the proof that the other cases can be reduced to that one.

Let $\tau_0 \in J$. It suffices to prove

$$|E(Iu(\tau_0)) - E(Iu(0))| \lesssim \frac{\mathcal{X}^4(\tau)}{N^4}$$

(5.1)

In what follows we also assume that $\tau_0 = \tau$: the reader can check after reading the proof that the other cases can be reduced to this one.

The Plancherel formula and the fundamental theorem of calculus yield

$$E(Iu(\tau)) - E(Iu(0)) = \int_0^\tau \int_{\xi_1 + \ldots + \xi_4 = 0} \mu(\xi_2, \xi_3, \xi_4) \hat{\partial_1 Iu}(t, \xi_1) \hat{Iu}(t, \xi_2) \hat{Iu}(t, \xi_3) \hat{Iu}(t, \xi_4) \, d\xi_2 d\xi_3 d\xi_4 \, dt$$

(5.2)

with

$$\mu(\xi_2, \xi_3, \xi_4) = 1 - \frac{m(\xi_2, \xi_3, \xi_4)}{m(\xi_2) m(\xi_3) m(\xi_4)}$$

(5.3)

We are left to prove

$$\left| \int_0^\tau \int_{\xi_1 + \ldots + \xi_4 = 0} \mu(\xi_2, \xi_3, \xi_4) \hat{\partial_1 Iu}(t, \xi_1) \hat{Iu}(t, \xi_2) \hat{Iu}(t, \xi_3) \hat{Iu}(t, \xi_4) \, d\xi_2 d\xi_3 d\xi_4 \, dt \right| \lesssim \frac{\mathcal{X}^4(\tau)}{N^4}$$

(5.4)

We perform a Paley-Littlewood decomposition to prove (5.4). Let $u_i = P_N u$ with $i \in \{1, \ldots, 4\}$ and let
(5.5) \[ X = \left| \int_0^\tau \int_{\xi_1,\ldots,\xi_4=0} \mu(\xi_2, \xi_3, \xi_4) \partial_{\xi_1} \widehat{Iu}_1(t, \xi_1) \widehat{Iu}_2(t, \xi_2) \widehat{Iu}_3(t, \xi_3) \widehat{Iu}_4(t, \xi_4) d\xi_2 d\xi_3 d\xi_4 dt \right| \]

There are different cases resulting from this Paley-Littlewood analysis and we describe now the strategy to estimate (5.4). We suggest that the reader at first ignores the second and third steps of the description and the \( N_j^\pm \) appearing in the study of these cases to solve the summation issue.

**Description of the strategy**

1. We follow [6] to estimate \( X \). First we recall the following Coifman-Meyer theorem [4], p179 for a class of multilinear operators

   **Theorem 5.1. ”Coifman Meyer multiplier theorem”** Consider an infinitely differentiable symbol \( \sigma : \mathbb{R}^{nk} \to \mathbb{C} \) so that for all \( \alpha \in N^{nk} \) there exists \( c(\alpha) \) such that for all \( \xi = (\xi_1, \ldots, \xi_k) \in \mathbb{R}^{nk} \)

   \[ \left| \partial^\alpha \sigma(\xi) \right| \leq \frac{c(\alpha)}{(1+|\xi|)^{2m}} \]

   Let \( \Lambda_\sigma \) be the multilinear operator

   \[ \Lambda_\sigma(f_1, \ldots, f_k)(x) = \int_{\mathbb{R}^{nk}} e^{ix \cdot (\xi_1, \ldots, \xi_k)} \sigma(\xi_1, \ldots, \xi_k) \hat{f}_1(\xi_1) \cdots \hat{f}_k(\xi_k) d\xi_1 \cdots d\xi_k \]

   Assume that \( q_j \in (1, \infty), j \in \{1, \ldots, k\} \) are such that \( \frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_k} \leq 1 \). Then there is a constant \( C = C(q, n, k, c(\alpha)) \) so that for all Schwarz class functions \( f_1, \ldots, f_k \)

   \[ \| \Lambda_\sigma(f_1, \ldots, f_k) \|_{L^q(\mathbb{R}^n)} \leq C \| f_1 \|_{L^{q_1}(\mathbb{R}^n)} \cdots \| f_k \|_{L^{q_k}(\mathbb{R}^n)} \]

   Then we proceed as follows. We seek a pointwise bound on the symbol

   \[ |\mu(\xi_2, \xi_3, \xi_4)| \leq B(N_2, N_3, N_4) \]

   We factor \( B = B(N_2, N_3, N_4) \) out of the right side of (5.5) and we are left to evaluate

   \[ B \int_0^\tau \int_{\mathbb{R}^{nk}} \Lambda_\#(\partial_{\xi_1} \widehat{Iu}_1(t), \widehat{Iu}_2(t), \widehat{Iu}_3(t))(\xi_4) \widehat{Iu}_4(t, \xi_4) d\xi_4 dt \]

   We notice that the multiplier \( \Lambda_\# \) satisfy the bound (5.6) and by the Plancherel theorem, Hölder inequality, theorem 5.1 and Bernstein inequalities we have

   \[ X \lesssim B \| \partial_{\xi_1} \widehat{Iu}_1 \|_{L^{p_1}(0, \tau)} \| \widehat{Iu}_2 \|_{L^{p_2}(0, \tau)} \cdots \| \widehat{Iu}_4 \|_{L^{p_4}(0, \tau)} \| D^{1-m_4} \widehat{Iu}_4 \|_{L^q(\mathbb{R}^n)} \]

   \[ \lesssim B N_1^{m_1} N_2^{m_2-1} \cdots N_4^{m_4-1} \| \partial_{\xi_1} D^{-m_4} \widehat{Iu}_1 \|_{L^{q_1}(0, \tau)} \| D^{1-m_4} \widehat{Iu}_2 \|_{L^{q_2}(0, \tau)} \cdots \| D^{1-m_4} \widehat{Iu}_4 \|_{L^{q_4}(0, \tau)} \]

   \[ \lesssim B N_1^{m_1} N_2^{m_2-1} \cdots N_4^{m_4-1} \| D^{1-m_4} \widehat{Iu}_4 \|_{L^q(\mathbb{R}^n)} \]
with \((p_j, q_j)\) such that \(p_j \in [1, \infty] \) and \(q_j \in (1, \infty)\) for \(j = \{1, \ldots, 4\}\), \(\sum_{j=1}^{4} \frac{1}{p_j} = 1\), \(\sum_{j=1}^{4} \frac{1}{q_j} = 1\), \((p_j, q_j)\) \(m_j\)-wave admissible for some \(m_j\) such that \(0 \leq m_j < 1\) and \(\frac{1}{q_j} + \frac{1}{q_j} = \frac{1}{2}\).

(2) The series must be summable. Therefore in some cases we might create \(N_k^\pm\) for some \(k\)’s by considering slight variations \((p_k^\pm, q_k^\pm) \in [1, \infty] \times (1, \infty)\) of \((p_k, q_k)\) that are \(m_k\) \(-\) wave admissible and such that \(\frac{1}{p_k^\pm} + \frac{1}{q_k^\pm} = \frac{1}{2}\).

For instance if we create slight variations \((p_2^+, q_2^-), (p_4^-, q_4^+)\) of \((p_2, q_2), (p_4, q_4)\) respectively we have

\[
\begin{align*}
\| I u_2 \|_{L_{t}^{p_2^+} L_{x}^{q_2^-}} & \lesssim N_2^\pm N_2^{m_2-1} \| D^{1-(m_2^-)} I u_2 \|_{L_{t}^{p_2^+} L_{x}^{q_2^-}} \\
\| I u_4 \|_{L_{t}^{p_4^-} L_{x}^{q_4^+}} & \lesssim N_4^\pm N_4^{m_4-1} \| D^{1-(m_4^+)} I u_4 \|_{L_{t}^{p_4^-} L_{x}^{q_4^+}}
\end{align*}
\]

and (5.11) becomes

(5.13) \[ X \lesssim BN_2^\pm N_4^\pm N_4^{m_4-1} N_2^{m_2-1} \cdots N_4^{m_4-1} Z^4(\tau) \]

(3) When we deal with low frequencies, i.e \(N_k < 1\) for some \(k \in \{1, \ldots, 4\}\) we might consider generating \(N_k^+\) by creating a variation \((2^+, \infty^-)\) of \((2, \infty^-)\). Such a task cannot be directly performed since we unfortunately have

\[
\| I u_k \|_{L_{t}^{p_k^2} L_{x}^{q_k^2}} \lesssim N_k^- \| D^{1-(1^-)} I u_k \|_{L_{t}^{p_k^2} L_{x}^{q_k^2}} \lesssim N_k^- Z(\tau)
\]

But we can indirectly create \(N_k^+\) by appropriately using Hölder in time inequality. Indeed if \(\epsilon > 0\), \(\epsilon' > 0\) and \(\epsilon'' > 0\) are such that \(\frac{\epsilon'}{2} = \frac{\epsilon'}{2} - \frac{\epsilon''}{3}\) we get from Bernstein inequalities, Hölder in time inequality and Sobolev homogeneous embedding

\[
\begin{align*}
\| I u_k \|_{L_{t}^{p_k^2} (\{0, \tau\}) L_{x}^{2^+}} & \lesssim N_k^\epsilon \| D^{-\epsilon'} I u_k \|_{L_{t}^{2^-} (\{0, \tau\}) L_{x}^{2^-}} \\
& \lesssim N_k^\epsilon \| D^{-\epsilon'} I u_k \|_{L_{t}^{2^-} (\{0, \tau\}) L_{x}^{2^-}} \\
& \lesssim N_k^- \epsilon^\epsilon' \| D^{-\epsilon'} + \epsilon'' I u_k \|_{L_{t}^{2^-} (\{0, \tau\}) L_{x}^{2^-}} \\
& \lesssim N_k^- \epsilon^\epsilon' \| D^{1-(1^-)} I u_k \|_{L_{t}^{2^-} (\{0, \tau\}) L_{x}^{2^-}} \\
& \lesssim N_k^- \epsilon^\epsilon' \tau^{2^+} Z(\tau)
\end{align*}
\]

We would like \(\epsilon'' > \epsilon'\). A quick computation show that it suffices that \(\epsilon' > 3\epsilon\). Letting \(\epsilon' = 5\epsilon\) we get

\[
\| I u_k \|_{L_{t}^{p_k^2} (\{0, \tau\}) L_{x}^{2^+}} \lesssim N_k^\epsilon \tau^{2^+} Z(\tau)
\]

Now if we choose \(\epsilon > 0\) so small that \(|\tau|^{2\epsilon} \leq 2\) we eventually get

(5.16) \[N_k^\epsilon \tau^{2^+} Z(\tau)\]
\begin{equation}
\|Iu_k\|_{L^2((0,\tau)) L^\infty_x} \lesssim N^+_k Z(\tau)
\end{equation}

For the remainder of the paper we say that we directly create $N^+_k$ if we directly use Bernstein inequality like in (5.12) or (5.14) and we say that we indirectly create $N^+_k$ if we also use Hölder in time inequality to get (5.17).

This ends the general description of the strategy.

Let us get back to the proof. By symmetry we may assume that $N_2 \geq N_3 \geq N_4$. There are several cases

- **Case 1**: $N >> N_2 \geq N_3$. In this case $X = 0$ since $\mu = 0$.

- **Case 2**: $N_2 \gtrsim N >> N_3$

    In this case we have

\begin{equation}
|\mu(\xi_2, \ldots, \xi_4)| \lesssim \frac{|\nabla m(\xi_2)||\xi_3+\xi_4|}{m(\xi_2)} \lesssim \frac{N_3}{N_2}
\end{equation}

We also get $N_3 \sim N_2$ from the convolution constraint $\xi_1 + \ldots + \xi_4 = 0$.

We assume that $N_4 \geq 1$. By (5.18) and by the Bernstein inequalities we have

\begin{equation}
X \lesssim \frac{N_4}{N_2} \|D \partial Iu_1\|_{L^6((0,\tau)) L^2_x} \|Iu_2\|_{L^6((0,\tau)) L^2_x} \|Iu_3\|_{L^6((0,\tau)) L^2_x} \|Iu_4\|_{L^6((0,\tau)) L^2_x} \\
\lesssim N_4^+ N_2^- N_3^+ N_2^- \|D \partial Iu_1\|_{L^6((0,\tau)) L^2_x} \|D \partial Iu_2\|_{L^6((0,\tau)) L^2_x} \|D \partial Iu_3\|_{L^6((0,\tau)) L^2_x} \|D \partial Iu_4\|_{L^6((0,\tau)) L^2_x} \\
\lesssim \frac{N_4^- N_2^-}{N_2^+} Z^4(\tau)
\end{equation}

after directly creating $N^+_1$ and $N^-_4$. If $N_4 < 1$ the proof is similar except that we indirectly create $N^+_4$ to get $X \lesssim \frac{N_4^- N_2^-}{N_2^+} Z^4(\tau)$. This makes the summation possible. We get (5.4) after summation.

- **Case 3**: $N_3 \gtrsim N >> N_4$

    In this case we have

\begin{equation}
|\mu(\xi_2, \ldots, \xi_4)| \lesssim \frac{m(\xi_1)}{m(\xi_2) m(\xi_3) m(\xi_4)}
\end{equation}

There are two subcases

- **Case 3a**: $N_1 \sim N_2$

    We assume that $N_4 \geq 1$. By (5.20) we have

\begin{equation}
X \lesssim \frac{N_4^-}{N^+_1} \|D \partial Iu_1\|_{L^6((0,\tau)) L^2_x} \|Iu_2\|_{L^6((0,\tau)) L^2_x} \|Iu_3\|_{L^6((0,\tau)) L^2_x} \|Iu_4\|_{L^6((0,\tau)) L^2_x} \\
\lesssim N_4^- N_1^+ N_2^- N_3^+ \|D \partial Iu_1\|_{L^6((0,\tau)) L^2_x} \|D \partial Iu_2\|_{L^6((0,\tau)) L^2_x} \|D \partial Iu_3\|_{L^6((0,\tau)) L^2_x} \|D \partial Iu_4\|_{L^6((0,\tau)) L^2_x} \\
\lesssim \frac{N_4^- N_2^-}{N_2^+} Z^4(\tau)
\end{equation}

after directly creating $N^+_1$ and $N^-_4$. If $N_4 < 1$ the proof is similar except that we indirectly create $N^+_4$. We get (5.4) after summation.
\[- \text{Case 3.b: } N_1 \ll N_2 \]
In this case by the convolution constraint $\xi_1 + \ldots + \xi_4 = 0$ we have $N_2 \sim N_3$. There are two subcases

* \text{Case 3.b.1: } N_1 \ll N

We assume that $N_1 \geq 1$ and $N_4 \geq 1$. By (5.20) we have

\[
X \lesssim \frac{N_1^{1/2}}{N_1^{1/2}} \| \partial_t Iu_1 \|_{L_t^2((0,\tau]) L_{\tau}^{1+}} \| Iu_2 \|_{L_t^2((0,\tau]) L_{\tau}^{1+}} \| Iu_3 \|_{L_t^2((0,\tau]) L_{\tau}^{1+}} \| Iu_4 \|_{L_t^2 ((0,\tau]) L_{\tau}^{2-}} \\
\| [D^{1-\frac{2}{3}} Iu_1 Iu_2 Iu_3]^3 \|_{L_{\tau}^{2+}} \| D^{1-\frac{1}{3}} Iu_4 \|_{L_{\tau}^{2+}} \| D^{1-\frac{1}{3}} Iu_4 \|_{L_{\tau}^{2+}} \\
\lesssim \frac{N_1^2 N_2^{1/2}}{N_1^{1/2}} \| \partial_t Iu_1 \|_{L_t^2((0,\tau]) L_{\tau}^{1+}} \| Iu_2 \|_{L_t^2((0,\tau]) L_{\tau}^{1+}} \| Iu_3 \|_{L_t^2((0,\tau]) L_{\tau}^{1+}} \\
\| D^{1-\frac{2}{3}} Iu_1 Iu_2 Iu_3 \|_{L_{\tau}^{2+}} \| D^{1-\frac{1}{3}} Iu_4 \|_{L_{\tau}^{2+}} \\
\lesssim \frac{N_1^2 N_2^{1/2}}{N_1^{1/2}} Z(\tau)
\]

after directly creating $N_1^+$ and $N_1^-$. If $N_1 < 1$ and $N_4 < 1$ the proof is similar except that we indirectly create $N_1^+$ and we substitute $N_1^-$ for $N_1^+$. The proof for the other cases \(^2\) is a slight variant to that for the case $N_1 \geq 1, N_4 \geq 1$ and that for the case $N_1 < 1, N_4 < 1$. Details are left to the reader. We get (5.4) after summation.

* \text{Case 3.b.2: } N_1 \geq N

We assume that $N_4 \geq 1$. By (5.20) we have

\[
X \lesssim \frac{N_1^{1/2}}{N_1^{1/2}} \| \partial_t Iu_1 \|_{L_t^2((0,\tau]) L_{\tau}^{1+}} \| Iu_2 \|_{L_t^2((0,\tau]) L_{\tau}^{1+}} \| Iu_3 \|_{L_t^2((0,\tau]) L_{\tau}^{1+}} \\
\| Iu_4 \|_{L_t^2 ((0,\tau]) L_{\tau}^{2-}} \\
\| D^{1-\frac{2}{3}} Iu_1 Iu_2 Iu_3 \|_{L_{\tau}^{2+}} \\
\| D^{1-\frac{1}{3}} Iu_4 \|_{L_{\tau}^{2+}} \\
\lesssim \frac{N_1^2 N_2^{1/2}}{N_1^{1/2}} Z(\tau)
\]

after directly creating $N_1^+$ and $N_1^-$. If $N_4 < 1$ the proof is similar except that we indirectly create $N_4^+$. We get (5.4) after summation.

* \text{Case 4: } N_4 \geq N

There are two subcases

* \text{Case 4.a: } N_1 \sim N_2

By (5.20) we have

\[
X \lesssim \frac{N_1^{1/2}}{N_1^{1/2}} \| \partial_t Iu_1 \|_{L_t^2((0,\tau]) L_{\tau}^{1+}} \| Iu_2 \|_{L_t^2((0,\tau]) L_{\tau}^{1+}} \| Iu_3 \|_{L_t^2((0,\tau]) L_{\tau}^{1+}} \\
\| Iu_4 \|_{L_t^2 ((0,\tau]) L_{\tau}^{2-}} \\
\| D^{1-\frac{2}{3}} Iu_1 Iu_2 Iu_3 \|_{L_{\tau}^{2+}} \\
\| D^{1-\frac{1}{3}} Iu_4 \|_{L_{\tau}^{2+}} \\
\lesssim \frac{N_1^2 N_2^{1/2}}{N_1^{1/2}} Z(\tau)
\]

\(^2\) i.e $N_1 \geq 1, N_4 \leq 1$ or $N_4 \leq 1, N_4 \geq 1$
after directly creating $N^-_2$ and $N^+_2$. We get (5.4) after summation.

- **Case 4.b**: $N_1 << N_2$

  In this case we have $N_2 \sim N_3$. There are two subcases

  * **Case 4.b.1**: $N_1 \gtrsim N$

    By (5.20) we have

    \[
    X \lesssim \frac{N^{2(1-s)} N_{1-s}^{1-s} N_{1-s}}{N_{2(1-s)}^{N_{2(1-s)}} N_{1-s}^{N_{1-s}}} \|\partial_t Iu_1\|_{L^4_t([0, \tau])} L^4 \|Iu_2\|_{L^4_t([0, \tau])} L^4 \|Iu_3\|_{L^4_t([0, \tau])} L^4
    \]

    We get (5.4) after summation.

  * **Case 4.b.2**: $N_1 \ll N$

    We assume that $N_1 \geq 1$. We have

    \[
    X \lesssim \frac{N^{2(1-s)} N_{1-s}^{1-s} N_{1-s}}{N_{2(1-s)}^{N_{2(1-s)}} N_{1-s}^{N_{1-s}}} \|\partial_t Iu_1\|_{L^4_t([0, \tau])} L^4 \|Iu_2\|_{L^4_t([0, \tau])} L^4 \|Iu_3\|_{L^4_t([0, \tau])} L^4
    \]

    If $N_1 < 1$ the proof is similar except that we create $N^+_1$ instead of $N^-_1$. We get (5.4) after summation.

6. **Proof of Almost Morawetz-Strauss inequality**

We prove proposition 2.4 in this section. The proof is divided into two steps

- **First Step**: Morawetz-Strauss inequality

  We recall the proof of the Morawetz-Strauss inequality ([12], [13]). We have the following identity

  \[
  \left( \frac{x \cdot u}{|x|^3} + \frac{u}{|x|^3} \right) (u_t - \Delta u + u^3) = \partial_t \left( \frac{x}{|x|^3} (x \cdot \nabla u + u) \partial_t u \right) + \nabla \left[ \frac{1}{|x|^3} \left( - \frac{1}{2} (\partial_t u)^2 - (x \cdot \nabla u) \nabla u + \frac{1}{2} |\nabla u|^2 x - u \nabla u - \frac{u^2}{2|x|^3} x + \frac{u}{4} u^4 x \right) \right] + \frac{1}{|x|^3} \left( |\nabla u|^2 - \frac{x \cdot u^3}{|x|^3} \right) + \frac{u^4}{|x|^3}
  \]

  and since $u$ satisfies (1.1) we have after integration
\[ 2\pi \int_0^T u^2(t,0) dt + \int_0^T \int_{\mathbb{R}^3} \frac{u^4(t,x)}{2|x|} \, dx \, dt \quad = \quad - \int_{\mathbb{R}^3} \left( \frac{\nabla u(T,x) \cdot x}{|x|} + \frac{u(T,x)}{|x|} \right) \partial_t u(T,x) \, dx \\
+ \int_{\mathbb{R}^3} \left( \frac{\nabla u(0,x) \cdot x}{|x|} + \frac{u(0,x)}{|x|} \right) \partial_t u(0,x) \, dx \]

\[(6.2)\]

Now we apply the basic inequality \(|ab| \leq \frac{|a|^2}{2} + \frac{|b|^2}{2}\) to the right hand side of the integral and we get

\[ \int_0^T \int_{\mathbb{R}^3} \frac{u^4(t,x)}{2|x|} \, dx \, dt \leq \frac{1}{2} \int_{\mathbb{R}^3} \left( \frac{\nabla u(T,x) \cdot x}{|x|} + \frac{u(T,x)}{|x|} \right)^2 + (\partial_t u)^2(T,x) \, dx \\
+ \frac{1}{2} \int_{\mathbb{R}^3} \left( \frac{\nabla u(0,x) \cdot x}{|x|} + \frac{u(0,x)}{|x|} \right)^2 + (\partial_t u)^2(0,x) \, dx \]

\[(6.3)\]

We also notice that

\[ \left( \frac{\nabla u \cdot x}{|x|} + \frac{u}{|x|} \right)^2 = \frac{(|\nabla u|^2)^2}{|x|^2} + div \left( \frac{u^2}{|x|^2} x \right) \]

\[ \leq |\nabla u|^2 + div \left( \frac{u^2}{|x|^2} x \right) \]

\[(6.4)\]

We plug (6.4) into (6.3). We get the Morawetz-Strauss’s inequality

\[ \int_0^T \int_{\mathbb{R}^3} \frac{u^4(t,x)}{2|x|} \, dx \, dt \quad \leq \quad 2 (E(u(T)) + E(u(0))) \]

- **Second Step**: Almost Morawetz-Strauss’s inequality. We substitute \( u \) for \( Iu \) in (6.1) and we proceed similarly. We get

\[ \int_0^T \int_{\mathbb{R}^3} \frac{|Iu|^4(t,x)}{|x|} \, dx \, dt - 2 (E(Iu(T)) + E(Iu(0))) \quad \leq \quad |R_1(T) + R_2(T)| \]

\[ \leq \quad |R_1(T)| + |R_2(T)| \]

### 7. Proof of the integral estimates

We are interested in proving proposition 2.5 in this section. In what follows we also assume that \( J = [0, \tau] \); the reader can check after reading the proof that the other cases can be reduced to that one.

Plancherel formula yields

\[ R_1(\tau) = \int_0^T \int_{\xi_1 + \ldots + \xi_4 = 0} \mu(\xi_2, \xi_3, \xi_4) \hat{Iu}(t, \xi_1) \hat{Iu}(t, \xi_2) \hat{Iu}(t, \xi_3) \hat{Iu}(t, \xi_4) d\xi_2 \ldots d\xi_4 \, dt \]

and

\[ R_2(\tau) = \int_0^T \int_{\xi_1 + \ldots + \xi_4} \mu(\xi_2, \xi_3, \xi_4) \hat{Iu}(t, \xi_1) \hat{Iu}(t, \xi_2) \hat{Iu}(t, \xi_3) \hat{Iu}(t, \xi_4) d\xi_2 \ldots d\xi_4 \, dt \]
with \( \mu \) defined in (5.3). It suffices to prove

\[
\left| \int_0^T \int_{\xi_1 + \ldots + \xi_4 = 0} \mu(\xi_2, \xi_3, \xi_4) \frac{\nabla \tilde{u}}{|x|} (t, \xi_1) \tilde{u}(t, \xi_2) \tilde{u}(t, \xi_3) \tilde{u}(t, \xi_4) \, d\xi_2 \ldots d\xi_4 \, dt \right| \lesssim \frac{\mathcal{Z}_4^{(1)}(r)}{N^1} \tag{7.3}
\]

and

\[
\left| \int_0^T \int_{\xi_1 + \ldots + \xi_4 = 0} \mu(\xi_2, \xi_3, \xi_4) \frac{\nabla \tilde{u}}{|x|} (t, \xi_1) \tilde{u}(t, \xi_2) \tilde{u}(t, \xi_3) \tilde{u}(t, \xi_4) \, d\xi_2 \ldots d\xi_4 \, dt \right| \lesssim \frac{\mathcal{Z}_4^{(1)}(r)}{N^1} \tag{7.4}
\]

We perform a Paley-Littlewood decomposition to prove (7.3) and (7.4). Let \( u_i := P_{N_i} u, i \in \{2, \ldots, 4\} \), \( \left( \frac{\nabla \tilde{u}}{|x|} \right)_1 := P_{N_1} \left( \frac{\nabla \tilde{u}}{|x|} \right) \) and \( \left( \frac{\nabla \tilde{u}}{|x|} \right)_1 := P_{N_1} \left( \frac{\nabla \tilde{u}}{|x|} \right) \).

\[
X_1 = \left| \int_0^T \int_{\xi_1 + \ldots + \xi_4 = 0} \mu(\xi_2, \xi_3, \xi_4) \left( \frac{\nabla \tilde{u}}{|x|} \right)_1 (t, \xi_1) \tilde{u}_2(t, \xi_2) \tilde{u}_3(t, \xi_3) \tilde{u}_4(t, \xi_4) \, d\xi_2 \ldots d\xi_4 \, dt \right| \tag{7.5}
\]

and

\[
X_2 = \left| \int_0^T \int_{\xi_1 + \ldots + \xi_4 = 0} \mu(\xi_2, \xi_3, \xi_4) \left( \frac{\nabla \tilde{u}}{|x|} \right)_1 (t, \xi_1) \tilde{u}_2(t, \xi_2) \tilde{u}_3(t, \xi_3) \tilde{u}_4(t, \xi_4) \, d\xi_2 \ldots d\xi_4 \, dt \right| \tag{7.6}
\]

Notice that by Bernstein inequality, Hölder inequality, Plancherel theorem and (1.21) we have

\[
\left\| \left( \frac{\nabla \tilde{u}}{|x|} \right)_1 \right\|_{L^{\infty}_{t} L^2_{x}} \lesssim N_1^{\dagger} \left\| \nabla \tilde{u} \right\|_{L^1_{t} L^2_{x}} \tag{7.7}
\]

and

\[
\left\| \left( \frac{\nabla \tilde{u}}{|x|} \right)_1 \right\|_{L^{\infty}_{t} L^2_{x}} \lesssim N_1^{\dagger} \left\| \nabla \tilde{u} \right\|_{L^1_{t} L^2_{x}} \tag{7.8}
\]

If \( p_j \in [1, \infty) \) and \( q_j \in (1, \infty), j \in \{2, \ldots, 4\} \) such that \( \frac{1}{(\infty - \infty)} + \sum_{j=2}^{4} \frac{1}{p_j} = 1, \frac{1}{\infty} + \sum_{j=2}^{4} \frac{1}{q_j} = 1, (p_j, q_j) \) - wave admissible for some \( m_j \) s such that \( 0 \leq m_j < 1 \) and \( \frac{1}{p_j} + \frac{1}{q_j} = \frac{1}{2} \) then we have by the methodology explained in the proof of proposition 2.3

\[
X_1 \lesssim B(N_2, N_3, N_4) \left\| \left( \frac{\nabla \tilde{u}}{|x|} \right)_1 \right\|_{L^{\infty}_{t} \cap (0, \tau)} L^2_{x} \left\| \tilde{u}_2 \right\|_{L^{p_2}_{t} (0, \tau)} L^{q_2}_{x} \ldots \left\| \tilde{u}_4 \right\|_{L^{p_4}_{t} (0, \tau)} L^{q_4}_{x} \tag{7.9}
\]

and

\[
X_2 \lesssim B(N_2, N_3, N_4) \left\| \left( \frac{\nabla \tilde{u}}{|x|} \right)_1 \right\|_{L^{\infty}_{t} \cap (0, \tau)} L^2_{x} \left\| \tilde{u}_2 \right\|_{L^{p_2}_{t} (0, \tau)} L^{q_2}_{x} \ldots \left\| \tilde{u}_4 \right\|_{L^{p_4}_{t} (0, \tau)} L^{q_4}_{x} \tag{7.10}
\]
By symmetry we can assume that \( N_2 \geq N_3 \geq N_4 \). There are different cases

- **Case 1:** \( N \gg N_2 \geq N_3 \). In this case \( X_1 = 0 \) and \( X_2 = 0 \) since \( \mu = 0 \).
- **Case 2:** \( N_2 \gg N \gg N_3 \) By (5.17), (5.18), (7.7) and (7.8) we have

\[
X_1 \lesssim \frac{N_2}{N_4} \left( \frac{\sum |Iu|}{|x|} \right) \left\| \frac{1}{L^\infty - (0, \tau)} \right\| L^2 \left\| \frac{1}{L^2} \right\| \left\| D^{1-(1-\mu)} \right\| L^{+2}_{\mu} \left\| D^{1-(1-\mu)} \right\| L^{2}_{\mu}
\]
\[
\lesssim \frac{N_2}{N_4} N_4^+ \left( \frac{1}{N_2} N_4^+ N_4^+ \right) \| D^{1-(1-\mu)} \| L^{+2}_{\mu} \left\| D^{1-(1-\mu)} \right\| L^{2}_{\mu}
\]
\[
\lesssim \frac{N_2}{N_4} N_4^+ \left( \frac{1}{N_2} N_4^+ N_4^+ \right) \| D^{1-(1-\mu)} \| L^{+2}_{\mu} \left\| D^{1-(1-\mu)} \right\| L^{2}_{\mu}
\]
\[
\lesssim \frac{N_2}{N_4} N_4^+ \left( \frac{1}{N_2} N_4^+ N_4^+ \right) \| D^{1-(1-\mu)} \| L^{+2}_{\mu} \left\| D^{1-(1-\mu)} \right\| L^{2}_{\mu}
\]

and

\[
X_2 \lesssim \frac{N_2}{N_4} \left( \frac{1}{|x|} \right) \left\| \frac{1}{L^\infty - (0, \tau)} \right\| L^2 \left\| \frac{1}{L^2} \right\| \left\| D^{1-(1-\mu)} \right\| L^{+2}_{\mu} \left\| D^{1-(1-\mu)} \right\| L^{2}_{\mu}
\]
\[
\lesssim \frac{N_2}{N_4} N_4^+ \left( \frac{1}{N_2} N_4^+ N_4^+ \right) \| D^{1-(1-\mu)} \| L^{+2}_{\mu} \left\| D^{1-(1-\mu)} \right\| L^{2}_{\mu}
\]
\[
\lesssim \frac{N_2}{N_4} N_4^+ \left( \frac{1}{N_2} N_4^+ N_4^+ \right) \| D^{1-(1-\mu)} \| L^{+2}_{\mu} \left\| D^{1-(1-\mu)} \right\| L^{2}_{\mu}
\]

- **Case 3:** \( N_3 \ll N \gg N_4 \)

There are two subcases:
- **Case 3.a:** \( N_4 \sim N_3 \)

By (5.17), (5.20) and (7.7)

\[
X_1 \lesssim \frac{N_4^{1-\mu}}{N_4} \left( \frac{1}{N_4} \right) \left\| \frac{1}{L^\infty - (0, \tau)} \right\| L^2 \left\| \frac{1}{L^2} \right\| \left\| D^{1-(1-\mu)} \right\| L^{+2}_{\mu} \left\| D^{1-(1-\mu)} \right\| L^{2}_{\mu}
\]
\[
\lesssim \frac{N_4^{1-\mu}}{N_4} \left( \frac{1}{N_4} \right) \left\| \frac{1}{L^\infty - (0, \tau)} \right\| L^2 \left\| \frac{1}{L^2} \right\| \left\| D^{1-(1-\mu)} \right\| L^{+2}_{\mu} \left\| D^{1-(1-\mu)} \right\| L^{2}_{\mu}
\]
\[
\lesssim \frac{N_4^{1-\mu}}{N_4} \left( \frac{1}{N_4} \right) \left\| \frac{1}{L^\infty - (0, \tau)} \right\| L^2 \left\| \frac{1}{L^2} \right\| \left\| D^{1-(1-\mu)} \right\| L^{+2}_{\mu} \left\| D^{1-(1-\mu)} \right\| L^{2}_{\mu}
\]

Similarly we get
\[
X_2 \lesssim \frac{N_4^{1-\mu}}{N_4} \left( \frac{1}{N_4} \right) \left\| \frac{1}{L^\infty - (0, \tau)} \right\| L^2 \left\| \frac{1}{L^2} \right\| \left\| D^{1-(1-\mu)} \right\| L^{+2}_{\mu} \left\| D^{1-(1-\mu)} \right\| L^{2}_{\mu}
\]

for \( X_2 \), \( \left( \frac{1}{|x|} \right) \) respectively in (7.13).
- **Case 3.b:** \( N_4 \ll N \)

There are two subcases
* **Case 3.b.1** \( N_4 \ll N \)

\[
X_1 \lesssim \frac{N_4^{2(1-\mu)}}{N_4} \left( \frac{1}{N_4} \right) \left\| \frac{1}{L^\infty - (0, \tau)} \right\| L^2 \left\| \frac{1}{L^2} \right\| \left\| D^{1-(1-\mu)} \right\| L^{+2}_{\mu} \left\| D^{1-(1-\mu)} \right\| L^{2}_{\mu}
\]
\[
\lesssim \frac{N_4^{2(1-\mu)}}{N_4} \left( \frac{1}{N_4} \right) \left\| \frac{1}{L^\infty - (0, \tau)} \right\| L^2 \left\| \frac{1}{L^2} \right\| \left\| D^{1-(1-\mu)} \right\| L^{+2}_{\mu} \left\| D^{1-(1-\mu)} \right\| L^{2}_{\mu}
\]
\[
\lesssim \frac{N_4^{2(1-\mu)}}{N_4} \left( \frac{1}{N_4} \right) \left\| \frac{1}{L^\infty - (0, \tau)} \right\| L^2 \left\| \frac{1}{L^2} \right\| \left\| D^{1-(1-\mu)} \right\| L^{+2}_{\mu} \left\| D^{1-(1-\mu)} \right\| L^{2}_{\mu}
\]

Similarly we get
\[
X_2 \lesssim \frac{N_4^{2(1-\mu)}}{N_4} \left( \frac{1}{N_4} \right) \left\| \frac{1}{L^\infty - (0, \tau)} \right\| L^2 \left\| \frac{1}{L^2} \right\| \left\| D^{1-(1-\mu)} \right\| L^{+2}_{\mu} \left\| D^{1-(1-\mu)} \right\| L^{2}_{\mu}
\]

* **Case 3.b.2** \( N_4 \gg N \)
We get (7.3) and (7.4) after summation.

\[ X \lesssim \sum \left( \frac{N_1^{1-s}}{N_1^{1-s}} \right) \left\| \sum_{\|\tau\| < 1} \left| Iu \right| \right\|_{L_t^1 L_x^\infty} \left\| Iu_2 \right\|_{L_t^2 L_x^2} \left\| Iu_3 \right\|_{L_t^2 L_x^2} \left\| Iu_4 \right\|_{L_t^2 L_x^\infty} \]
\[ \lesssim \sum \left( \frac{N_1^{1-s} N_1^{1-s} N_1^{1-s} N_1^{1-s}}{N_1^{1-s} N_1^{1-s} N_1^{1-s} N_1^{1-s}} \right) \left\| DIu \right\|_{L_t^\infty L_x^2} \left\| DIu_2 \right\|_{L_t^\infty L_x^2} \left\| D(1-1)Iu_3 \right\|_{L_t^2 L_x^2} \left\| D(1-1)Iu_4 \right\|_{L_t^2 L_x^\infty} \]
\[ \lesssim \sum \left( \frac{N_1^{1-s} N_1^{1-s} N_1^{1-s} N_1^{1-s}}{N_1^{1-s} N_1^{1-s} N_1^{1-s} N_1^{1-s}} \right) Z^4(\tau) \]

Similarly \( X_2 \lesssim \frac{N_2^{1-s} N_1^{1-s} N_1^{1-s}}{N_1^{1-s} N_1^{1-s}} Z^4(\tau) \).

\* Case 4: \( N_4 \geq N \)
There are two subcases
- Case 4.a: \( N_1 \sim N_2 \)

We get (7.3) and (7.4) after summation.

\[ X \lesssim \sum \left( \frac{N_1^{1-s} N_1^{1-s} N_1^{1-s} N_1^{1-s}}{N_1^{1-s} N_1^{1-s} N_1^{1-s} N_1^{1-s}} \right) \left\| \sum_{\|\tau\| < 1} \left| Iu \right| \right\|_{L_t^1 L_x^\infty} \left\| Iu_2 \right\|_{L_t^2 L_x^2} \left\| Iu_3 \right\|_{L_t^2 L_x^2} \left\| Iu_4 \right\|_{L_t^2 L_x^\infty} \]
\[ \lesssim \sum \left( \frac{N_1^{1-s} N_1^{1-s} N_1^{1-s} N_1^{1-s}}{N_1^{1-s} N_1^{1-s} N_1^{1-s} N_1^{1-s}} \right) \left\| DIu \right\|_{L_t^\infty L_x^2} \left\| DIu_2 \right\|_{L_t^\infty L_x^2} \left\| D(1-1)Iu_3 \right\|_{L_t^2 L_x^2} \left\| D(1-1)Iu_4 \right\|_{L_t^2 L_x^\infty} \]
\[ \lesssim \sum \left( \frac{N_1^{1-s} N_1^{1-s} N_1^{1-s} N_1^{1-s}}{N_1^{1-s} N_1^{1-s} N_1^{1-s} N_1^{1-s}} \right) Z^4(\tau) \]

Similarly \( X_2 \lesssim \frac{N_2^{1-s} N_1^{1-s} N_1^{1-s}}{N_1^{1-s} N_1^{1-s}} Z^4(\tau) \).

We get (7.3) and (7.4) after summation.

**References**


University of California, Los Angeles