SCATTERING OF ROUGH SOLUTIONS OF 3D NLKG WITH GENERAL DATA

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Abstract. We prove scattering of solutions below the energy norm of the 3D Klein-Gordon equation

\[ \begin{cases} 
\partial_t u - \Delta u + u = -|u|^{p-1}u \\
u(0, x) = u_0(x) \\
\partial_t u(0, x) = u_1(x) 
\end{cases} \]

with \(3 < p < 5\): this result extends those obtained in the energy class [5, 15, 16] and those obtained below the energy norm under the additional assumption of spherical symmetry [20]. In order to do that, we prove exponential-type decay estimates in \(H^s \times H^{s-1}\), \(s < 1\), by means of concentration [1] and a low-high frequency decomposition [2, 3, 7]: this is the starting point to prove scattering. On low frequencies we modify the arguments in [15, 16]; on high frequencies we use the smoothing effect of the solutions to control the error terms: this, combined with an almost conservation law, allows to prove these decay estimates.

1. Introduction

In this paper we consider the \(p\)-defocusing Klein-Gordon equation on \(\mathbb{R}^3\)

\[(1.1) \quad \partial_t u - \Delta u + u = -|u|^{p-1}u \]

with data \(u(0) = u_0, \partial_t u(0) = u_1\) lying in \(H^s, H^{s-1}\) respectively. Here \(H^s\) is the standard inhomogeneous Sobolev space i.e \(H^s\) is the completion of the Schwartz space \(\mathcal{S}(\mathbb{R}^3)\) with respect to the norm \(\|f\|_{H^s} := \|\langle D \rangle^s f\|_{L^2(\mathbb{R}^3)}\) where \(\langle D \rangle\) is the operator defined by \(\langle D \rangle f(\xi) := (1 + |\xi|^2)^{\frac{1}{2}} \hat{f}(\xi)\) and \(\hat{f}\) denotes the Fourier transform, i.e \(\hat{f}(\xi) := \int_{\mathbb{R}^3} f(x) e^{-ix \cdot \xi} dx\).

We are interested in the strong solutions of the \(p\)-defocusing Klein-Gordon equation on some interval \([0, T]\) i.e maps \(u, \partial_t u\) that lie in \(C([0, T], H^s(\mathbb{R}^3)), C([0, T], H^{s-1}(\mathbb{R}^3))\) respectively and that satisfy

\[(1.2) \quad u(t) = \cos(t\langle D \rangle)u_0 + \frac{\sin(t\langle D \rangle)}{t\langle D \rangle}u_1 - \int_0^t \sin\left(\frac{(t-t')\langle D \rangle}{t\langle D \rangle}\right) |u|^{p-1}(t')u(t') dt' \]

The \(p\)-defocusing Klein-Gordon equation is closely related to the \(p\)-defocusing wave equation i.e

\[(1.3) \quad \partial_t v - \Delta v = -|v|^{p-1}v \]

with data \(v(0) = v_0, \partial_t v(0) = v_1\). (1.3) enjoys the following scaling property
We define the critical exponent \( s_c := \frac{\frac{3}{2} - \frac{2}{p+1}}{2} \). One can check that the \( \dot{H}^{s_c} \times \dot{H}^{s_c-1} \) norm of \((u_0, u_1)\) is invariant under the transformation \eqref{transformation}. \eqref{wave_eq} is known to be locally well-posed in \( H^s \times H^{s-1} \), \( s \geq \frac{\frac{3}{2} - \frac{2}{p+1}}{2} \), \( p \geq \frac{2}{3} \) by using an iterative argument.

If \( p = 5 \) then \( s_c = 1 \) and we say that that the nonlinearity \( |u|^{p-1}u \) is \( \dot{H}^1 \) (or energy) critical. If \( p = \frac{7}{3} \) then \( s_c = 0 \) and we say that the nonlinearity \( |u|^{p-1}u \) is \( L^2 \) (or mass) critical. If \( p = 3 \) then \( s_c = \frac{1}{2} \) and we say that the power is conformal. If \( \frac{7}{3} < p < 5 \) then we say that the regime is mass supercritical-energy subcritical. If \( 3 < p < 5 \) then we say that the regime is superconformal and energy subcritical.

It is well-known that smooth solutions to \eqref{wave_eq} have a conserved energy

\[
E(u(t)) := \frac{1}{2} \int_{\mathbb{R}^3} |\partial_t u(t,x)|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u(t,x)|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^3} |u(t,x)|^2 \, dx
\]

\[
+ \frac{1}{p+1} \int_{\mathbb{R}^3} |u(t,x)|^{p+1} \, dx
\]

In fact by standard limit arguments the energy conservation law remains true for solutions \((u, \partial_t u) \in H^s \times H^{s-1}, \ s \geq 1\).

Since the lifespan of the local solution depends only on the \( H^s \times H^{s-1} \) norm of the initial data \((u_0, u_1)\) (see \[11\]) for \( s > s_c \), then it suffices to find an a priori pointwise in time bound in \( H^s \times H^{s-1} \) of the solution \((u, \partial_t u)\) in order to establish global well-posedness. The energy captures the evolution in time of the \( H^1 \times L^2 \) norm of the solution. Since it is conserved we have global existence of \( H^1 \times L^2 \) solutions of \eqref{wave_eq} in the mass supercritical-energy subcritical regime.

The scattering theory (i.e the linear asymptotic behaviour of the solution of \eqref{wave_eq} in the energy space (i.e with data \((u_0, u_1) \in H^1 \times L^2\)) has attracted much attention from the community. In particular scattering was proved in \[4, 5, 9, 15, 16\] for all dimension \( n \) and for all exponent \( p \) that is \( H^1 \) subcritical and \( L^2 \) supercritical, i.e \( 1 < \frac{2}{3} < p < 1 + \frac{4}{n-2} \).

The global well-posedness theory below the energy norm in the subcritical regime (i.e for data in \( H^s \times H^{s-1}, \ s_c \leq s < 1 \)) has been investigated in \[19\]. In particular, it was proved that the solution exists globally in time for \( 1 < s > \bar{s} \), where \( \bar{s} \) is a fixed constant \( \bar{s} := \bar{s}(p) > s_c \) depending only on \( p \).

It remains to understand the scattering theory below the energy norm. It has recently been studied in dimension 3 and for radial data \[20\]. More precisely it was proved that the asymptotic behavior for spherical solutions is linear for \( 3 < p < 5 \) and in \( H^s \times H^{s-1} \), \( \bar{s} := \bar{s}(p) < s < 1 \) with

\[
\bar{s} := \begin{cases} 
1 - \frac{(5-p)(p-3)}{2(p-1)(p-2)}, & 4 \geq p > 3 \\
1 - \frac{(5-p)^2}{4(p-1)(6-p)}, & 5 \geq p \geq 4
\end{cases}
\]

\footnote{Here \( H^m \) denotes the standard homogeneous Sobolev space endowed with the norm \( \|f\|_{H^m} := \|D^m f\|_{L^2(\mathbb{R}^3)} \).}
In this paper we are interested in proving scattering results for general data below the energy norm and in dimension 3. The main result of this paper is the following one:

**Theorem 1.1.** Let $A \gtrsim 1^2$ and let $(u_0, u_1) \in H^s \times H^{s-1}$ such that

$$\| (u_0, u_1) \|_{H^s \times H^{s-1}} \leq A$$

Then there exist $\vec{\alpha} := (\alpha_1, \alpha_2, \alpha_3, \alpha_4) := (\alpha_1(p), \alpha_2(p), \alpha_3(p), \alpha_4(p)) \in [0, \infty)^4$, $\vec{C} := (C_1, C_2, C_3, C_4, C_5) := (C_1(p), C_2(p), C_3(p), C_4(p), C_5(p)) \in (\mathbb{R}^+)^5$ and $\vec{C} := \tilde{C}(p) \in \mathbb{R}^+$ such that the solution of the $p$-defocusing Klein-Gordon equation on $\mathbb{R}^3$ with data $(u_0, u_1)$ exists in $H^s \times H^{s-1}$ for all time $T$ and scatters as $T$ goes to infinity, i.e there exists $(u_{+,0}, u_{+,1}) \in H^s \times H^{s-1}$ such that

$$\lim_{T \to \infty} \| (u(T), \partial_t u(T)) - K(T)(u_{+,0}, u_{+,1}) \|_{H^s \times H^{s-1}} = 0$$

if $3 < p < 5$ and

$$1 > s > \bar{s}(A, p), \ |\bar{s} - 1| \sim_{A \to \infty} \frac{1}{X_{\vec{C}, \vec{\alpha}}^\gamma}$$

Moreover

$$\| Iu \|_{L_t^{2(p-1)} - L_t^{2(p-1)}(\mathbb{R})} \lesssim_A 1$$

Here

- $X_{\vec{C}, \vec{\alpha}} := Z_0$ where $(Z_p)_{0 \leq p \leq P}$ is a sequence defined as follows

$$\left\{ \begin{array}{l}
Z_p := e^{C_1 \max \left( 2C_2 z_p^{C_3 p+1} A^{\gamma_1} e^{C_4 A^{\gamma_2}} \right)^{C_5 A^{\gamma_3}}}
\end{array} \right.$$ 

with $P \sim A^{\gamma_1}$,

- $|\bar{s} - 1| \sim_{A \to \infty} \frac{1}{X_{\vec{C}, \vec{\alpha}}^\gamma}$ means that there exists $\gamma_1 := \gamma_1(p) > 0$ such that

$$|\bar{s} - 1| \frac{X_{\vec{C}, \vec{\alpha}}^\gamma}{\gamma_1} \to_{A \to \infty} 1$$

- $K(T) := \begin{pmatrix}
\cos(T(D)) & \sin(T(D)) \\
-\langle D \rangle \sin(T(D)) & \cos(T(D))
\end{pmatrix}$

**Remark 1.2.** We have also chosen not to find the optimal $((\vec{C}, \vec{\alpha}, \tilde{C})$. We could have done it but this would have complicated much more the reading of the proof of Theorem 1.1 that is technical.

We set some notation that appear throughout the proof.

We denote by $W$ the set of wave-admissible points, i.e

$$W := \left\{ (q, r), (q, r) \in (2, \infty] \times [2, \infty], \frac{1}{q} + \frac{1}{r} \leq \frac{1}{2} \right\}$$

Given $m \in [0, 1]$, we say that $(q, r)$ is $m$-wave admissible if

- $(q, r) \in (2, \infty] \times [2, \infty)$, $\frac{1}{q} + \frac{1}{r} \leq \frac{1}{2}$ and $\frac{1}{q} + \frac{3}{r} = \frac{3}{2} - m$ for $m \in [0, 1]$
\begin{itemize}
  \item $q > 2^+$ for $m = 1$
\end{itemize}
and we denote by $W_m$ the set of $m$-wave admissible points.

Let $I$ be the multiplier defined by $If(\xi) := \eta(\frac{\xi}{|\xi|}) \; \eta$ a smooth, radial, nonincreasing function in $|\xi|$ such that $\eta(\xi) := 1$, $|\xi| \leq 1$ and $\eta(\xi) := \frac{1}{|\xi|}$, $|\xi| \geq 2$.

If $f$ is differentiable in time and a smooth function in space then

\[ Z_{m,s}(J, f) := \sup_{\text{wave adm}} \| \partial_t(D)^{-m}If\|_{L^s_t L^r_x(J)} + \|(D)^{1-m}If\|_{L^s_t L^r_x(J)} \]
and $Z(J, f) := \sup_{m \in [0,1]} Z_{m,s}(J, f)$. We write $F(f)$ for the following function $F(f) := \|f\|^{p-1}f$. If $B(x_0, R) := \{x \in \mathbb{R}^3, |x - x_0| \leq R\}$ then

\[ E(If(t)) := \frac{1}{2} \int_{\mathbb{R}^3} |\partial_x If(t, x)|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} |If(t, x)|^2 dx + \frac{1}{p+1} \int_{\mathbb{R}^3} |If(t, x)|^{p+1} dx \]

\[ E_c(If(t)) := \frac{1}{2} \int_{\mathbb{R}^3} |\partial_x If(t, x)|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} |If(t, x)|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} |\nabla If(t, x)|^2 dx \]
and

\[ E(If(t), B(x_0, R)) := \frac{1}{2} \int_{B(x_0, R)} |\partial_x If(t, x)|^2 dx + \frac{1}{2} \int_{B(x_0, R)} |If(t, x)|^2 dx + \frac{1}{p+1} \int_{B(x_0, R)} |If(t, x)|^{p+1} dx \]

Some estimates that we establish throughout the paper require a Paley-Littlewood decomposition. We set it up now. Let $\phi(\xi)$ be a real, radial, nonincreasing function that is equal to 1 on the unit ball $\{\xi \in \mathbb{R}^3 : |\xi| \leq 1\}$ and that is supported on $\{\xi \in \mathbb{R}^3 : |\xi| \leq 2\}$. Let $\psi$ denote the function $\psi(\xi) := \phi(\xi) - \phi(2\xi)$. If $(M, M_1, M_2) \in 2^N$ are dyadic numbers such that $M_2 \geq M_1$ we define the Paley-Littlewood operators in the Fourier domain by

\[ \tilde{P}_M f(t, \xi) := \phi\left( \frac{\xi}{M} \right) \hat{f}(t, \xi), \quad M > 1 \]
\[ \tilde{P}_1 f(t, \xi) := \phi(\xi) \hat{f}(t, \xi) \]
\[ \tilde{P}_{\leq M} f(t, \xi) := \phi\left( \frac{\xi}{M} \right) \hat{f}(t, \xi) \]
\[ \tilde{P}_{>M} f(t, \xi) := \hat{f}(t, \xi) - \tilde{P}_{\leq M} f(t, \xi) \]
\[ \tilde{P}_{<M} f(t, \xi) := \tilde{P}_{\leq M} f(t, \xi) \]
\[ \tilde{P}_{\leq M_2} f := \tilde{P}_{\geq M_2} f - \tilde{P}_{<M_1} f \]

Since $\phi(\xi) + \sum_{M \in 2^{[0,\infty) \setminus \{0\}}} \psi\left( \frac{\xi}{M} \right) = 1$ we have $f(t) = \sum_{M \in 2^N} P_M f(t)$. Notice also that $f(t) = P_{<M} f(t) + P_{\geq M} f(t)$.

Let $w$ be a solution of (1.1). Let $J := [a, b] \subset [0, \infty)$. Let $J' \subset J$. Given $T \in J$ and $v$, a solution of the free Klein-Gordon equation, we let $w^1$ be the solution of (1.1) with the same initial data as $w - v$ at $t = T$.

Throughout this proof $\alpha := \alpha(p) > 0$ and $C := C(p) > 0$ denote two constants that do only depend on $p$, whose value is allowed to change
• from one line to the other one. For example (5.19) and (5.22) mean that there exist \( \alpha_1 := \alpha_1(p) > 0 \) and \( \alpha_2 := \alpha_2(p) > 0 \) such that

\[
|J'| \gtrsim (A^{\frac{p+1}{2}} N^{\frac{1}{s}})^{\alpha_1}
\]

and

\[
\langle M \rangle \lesssim (A^{\frac{p+1}{2}} N^{1-s})^{\alpha_2}
\]
• in the same expression. For example (2.15) means that there exist \((C_1, C_2, C_3, \alpha_1, \alpha_2) := (C_1(p), C_2(p), C_3(p), \alpha_1(p), \alpha_2(p)) \in [0, \infty]^5 \) and such that

\[
\|w\|_{L^2_t L^2_x([0,T])} \geq e^{C_1 \max (M, e^{C_2(N^{1-s} A^{\frac{p+1}{2}})})} \]

and such that all the estimates are true. Given two positive numbers \( x \) and \( y \) depending only on \( p \), we say that \( x \sim y \) if there exists \( \gamma := \gamma(p) > 0 \) such that \( \frac{1}{\gamma} y \leq x \leq \gamma y \). We say that \( x = o(y) \) if there exists \( 0 < \gamma := \gamma(p) \ll 1 \) such that \( x \leq \gamma y \). Given two positive numbers \( x \) and \( y \) depending on \( p \) and \( A \), we say that \( x \sim A \to \infty y \) if there exists \( \gamma := \gamma(p) > 0 \) such that \( x \gamma y \to A \to \infty 1 \).

Now we recall some propositions that we constantly use throughout this proof.

The wave Strichartz estimates (see for example [20]) \(^3\) can be stated as follows:

**Proposition 1.3. ”Wave Strichartz estimates”** Assume that \( u \) satisfies the following Klein-Gordon equation on \( \mathbb{R}^3 \)

\[
\begin{aligned}
\partial_t u - \Delta u + u &= G \\
u(0,x) &= u_0(x) \\
\partial_t u(0,x) &= u_1(x)
\end{aligned}
\]

Let \( T \geq 0 \). Then, if \( m \in [0,1] \)

\[
\begin{aligned}
\|u\|_{L^2_t L^2_x([0,T])} + \|\partial(D)^{-1}u\|_{L^2_t L^2_x([0,T])} + \|u\|_{L^\infty_x H^m([0,T])} + \|\partial u\|_{L^2_x H^{m-1}([0,T])}
\lesssim \|u_0\|_{H^m} + \|u_1\|_{H^{m-1}} + \|G\|_{L^2_t L^2_x([0,T])}
\end{aligned}
\]

under the following assumptions

• \((q, r) \in W_m\)

• \((\tilde{q}, \tilde{r}) \) lies in the dual set \( \tilde{W} \) of \( W \) defined by

\[
\tilde{W} := \left\{ (x, y), \exists (x, y) \in W, \frac{1}{x} + \frac{1}{y} = 1, \frac{1}{x} + \frac{1}{\tilde{y}} = 1 \right\}
\]

and it satisfies the following inequality

\(^3\)Notice that there are more complicated Strichartz estimates in Besov spaces: we refer to [14] for more details.
\[ \frac{1}{q} + \frac{3}{r} - 2 = \frac{1}{q} + \frac{3}{r} \]

The next proposition yields an upper bound of the mollified energy at time 0:

**Proposition 1.4.** "Mollified energy at time 0 is bounded by \( N^2(1-s)A^{p+1} \)"

[20] There exists \( C_E > 0 \) such that

\[
E(Iu_0) \leq C_E N^2(1-s)A^{p+1}
\]

The next proposition shows that the variation of the mollified energy of \( w \) can be controlled on an arbitrary time interval \( K \) by the parameter \( N \) and the number \( Z(K,w) \):

**Proposition 1.5.** "Almost Conservation Law" [20] Let \( s \geq \frac{3p-5}{2p} \). Let \( t_0 \in K \). Then

\[
|\sup_{t \in K} E(Iw(t)) - E(Iw(t_0))| \lesssim \left| \int_K \int_{\mathbb{R}^3} \mathbb{R}(\partial_t Iw(IF(w) - F(Iw))) \, dx \, dt \right| \lesssim \frac{Z^{p+1}(K,w)}{N^{\frac{s}{2p}}}
\]

The last two propositions, when they are combined together, allow to control a weighted norm of \( w \) on \( K \):

**Proposition 1.6.** "Estimate of integrals" [20] Assume that \( s \geq \frac{3p-5}{2p} \). Then for \( i = 1, 2 \) we have

\[
|R_i(K, w)| \lesssim \frac{Z^{p+1}(K,w)}{N^{\frac{s}{2p}}}
\]

with \( R_1(K, w) := \int_K \int_{\mathbb{R}^3} \frac{\nabla Iw(t,x) \cdot (F(Iw) - IF(w))}{|x|} \, dx \, dt \)

and

\[
R_2(K, w) := \int_K \int_{\mathbb{R}^3} \frac{\nabla Iw(t,x) \cdot (F(Iw) - IF(w))}{|x|} \, dx \, dt.
\]

**Proposition 1.7.** "Almost Morawetz-Strauss Estimate" [20] Let \( x_0 \in \mathbb{R}^3 \) and let \( T \geq 0 \). Then

\[
\int_0^T \int_{\mathbb{R}^3} \frac{|Iw(t,x)|^{p+1}}{|x-x_0|} \, dx \, dt \lesssim \sup_{t \in [0, T]} E(Iw(t)) + R_1([0, T], w) + R_2([0, T], w)
\]

Now we explain the main ideas of this proof.

In [15, 16], Nakanishi found an upper bound of the \( L^q_L^r \) norm of the solution of (1.1) with data in the energy class (i.e \((u_0, u_1) \in H^1 \times L^2\)) by a tower-exponential type bound of the energy and for some \((q := q(p), r := r(p))\): this decay estimate was the preliminary step to prove scattering. A natural question is: is it possible to
prove decay estimates of this form (or a modified form) in the subcritical case and for rough data (i.e. for data \((u_0, u_1) \in H^s \times H^{s-1}, s_0 < s < 1\) ? If this is possible, then it might help us to prove scattering of solutions of (1.1) with rough data, by analogy with the scattering theory for data in the energy space. This paper gives a positive answer to this question, at least for \(s\) close enough to one.

Of course, one cannot use the energy conservation law because the energy can be infinite on \(H^s \times H^{s-1}, s < 1\). Instead we work with a mollified energy \(E(Iu(t))\) that is finite in these rough spaces (see Proposition 1.4) and that is expected to be almost conserved \(^4\): this is the \(I\)-method, introduced first in [7] to study global existence of rough solutions of semilinear Schrödinger equations and inspired by the Fourier truncation method, designed in [3]. Since the purpose of the multiplier \(I\) is to make the energy finite, we aim at proving decay estimates involving not only \(u\) but also \(I\). It was proved in [20] that, under the additional assumption of radial symmetry, we can control pretty easily \(\|Iu\|_{L^{p+2}_t L^{s+2}_x(\mathbb{R})}\) by combining the “Almost Morawetz-Strauss estimate” (1.13) with a pointwise decay estimate around the origin (a radial Sobolev inequality):

\[
|Iu(t, x)| \lesssim \frac{\|Iu(t, x)\|_{H^1}}{|t|^{\frac{1}{2}}},
\]

Unfortunately, there is no useful pointwise estimate to our knowledge in the general case and we shall prove, by means of concentration [1], that norms of the form \(\|Iu\|_{L^{p+2}_t L^{s+2}_x(\mathbb{R})}\) are bounded. Now we explain how this paper is organized.

As mentioned above, we aim at proving exponential-type decay estimates of the form \(\|Iu\|_{L^{p+2}_t L^{s+2}_x(\mathbb{R})}\). In order to prove these estimates the idea is to use the \(H^1\) theory for frequencies smaller or equal to the parameter \(N\) and to control for frequencies larger than \(N\) all the errors that are created by the multiplier \(I\) and that appear in the process of generating these estimates. These bounds should not depend on time \(T\), if not it would kill the scattering: this can be done by letting the estimates only depend on the parameter \(N\). If we neglect, in a first approximation, the variation of the mollified energy in order to estimate the norm \(\|Iu\|_{L^{p+2}_t L^{s+2}_x(\mathbb{R})}\), we expect to find, by using (1.10), a tower-type exponential bound of this norm, by analogy with the bound of the \(L^2_t L^4_x\) norm of the solutions of (1.1) with data \((u_0, u_1) \in H^1 \times L^2\), say

\[
\|Iu\|_{L^{p+2}_t L^{s+2}_x(\mathbb{R})} \lesssim (CEN^{2(1-s)}A^{p+1})CEN^{2(1-s)}A^{p+1}.
\]

Then, combining this bound with the polynomial decay \(\left(\frac{1}{N^{s+2}}\right)\) of the almost conservation law we expect to find after iteration of the local theory (see Proposition 2.1) an error of the form (see Proposition 1.5)

\[
(CN^{1-s}A^{p+1})^{p+1} + (CEN^{2(1-s)}A^{p+1})^{p+1} + (CEN^{2(1-s)}A^{p+1})^{p+1},
\]

and it seems at first sight that there do not exist \(s_0 := s_0(p) < 1\) and \(N := N(A, p, s) >> 1\) such that we can make the error smaller than the main term \(CEN^{2(1-s)}A^{p+1}\) (see Proposition 1.4) of the mollified energy at time zero for \(s \geq s_0\),

\(^4\)since the multiplier \(I\) approaches the identity operator as \(N\) goes to infinity
since the denominator grows polynomially whereas the numerator grows exponentially. Notice, however, that

\[
\lim_{s \to 1^+} \left( \frac{CN^{1-s} A^{p+1}}{E} \right)^{p+1} \left( \frac{C_E N^{2(1-s)} A^{p+1} \eta C E N^{2(1-s)} A^{p+1}}{E} \right)^{p+1} = \frac{C E N^{2(1-s)} A^{p+1} N \eta}{C E A^{p+1}}
\]

and therefore, choosing \( N := N(A, p) >> 1 \) large enough, we see that there exists \( s_1 := s_1(A, p) < 1 \) such that the error is small compared with the main term for \( s_1 \leq s < 1 \): this is why the range of \( s \) for which there is scattering is not only to depend on \( p \), but also on the size of the initial data (i.e., \( A \)): see Theorem 1.1.

The \( H^1 \) theory is based upon the induction on levels of the conserved energy due to Bourgain [1]. Here we cannot expect the mollified energy to be conserved. This implies that we have to modify significantly the induction on energy method in order to establish a finite bound of \( \| I u \|_{L^2_x(t)} \). In particular, we design a relation that allows us to control not only \( \| I u \|_{L^2_x(t)} \) but also \( \sup_t E(Iu(t)) \) (with \( w \) solution of (1.1)), assuming that we control the mollified energy of \( w \) at one time (see the definition of \( P(l) \) in Section 2), and our goal is to prove that this relation is true for large levels of mollified energy at one time by induction, using the small mollified energy at one time theory (see Proposition 2.4). Also, one has to make sure that the methods of concentration, combined with the Almost Morawetz-Strauss estimate, allow to make the mollified energy \( E(Iu) \) decrease at one time at a non-decreasing rate if the \( L^p_x(t) \) norm of the \( Iu \) is large, in order to reach the small mollified energy level and apply the small mollified energy theory. In order to do that, we have to make the variation of the mollified energy small enough so that we can construct a solution \( w' \) of which the mollified energy at one time is smaller than \( \sup_t E(Iu(t)) \) by a nontrivial amount. This is why we introduce the parameter \( \beta \): see (2.30) for its use. The remainder of the argument is based upon a modification of arguments in [15, 16], modulo errors.

2. Proof of Theorem 1.1

In this section, we prove Theorem 1.1.

We shall also assume that some propositions (namely Proposition 2.1, 2.2 and 2.3), to be listed shortly and to be proved in the next sections, are true, provided that two types of assumptions are satisfied:

• \( a \) \textit{priori assumptions}: we will prove shortly by a bootstrap argument that these \( a \) \textit{priori} assumptions are in fact true. There are four numbers \( 0 \leq X_w < \infty, 0 \leq X_w' < \infty, 1 \leq L_w < \infty \) and \( 1 \leq L_w' < \infty \) such that

\[
\sup_{t \in I} E(Iu(t)) \leq X_w^2 \lesssim N^{2(1-s)} A^{p+1}.
\]

\(^5\)The constraints \( L_w \geq 1 \) and \( L_w' \geq 1 \) are not essential. They are imposed to simplify the computations. Indeed later we will often divide an arbitrarily long time interval in a partition of subintervals such that \( \| I u \|_{L^2_x(t)} \leq L_w \) is small. The number of these subintervals will be roughly speaking bounded by \( L_w^2 \) instead of \( \max(L_w^2, 1) \).
\[ (2.2) \quad \sup_{t \in J} E(Iw'(t)) \leq N_{w'}^2 \lesssim N^{2(1-s)}A^{p+1} \]

\[ (2.3) \quad ||Iw||_{L^2(p-1)-L^2(p-1)+ (J)} \leq L_\omega \]

and

\[ (2.4) \quad ||Iw'||_{L^2(p-1)-L^2(p-1)+ (J)} \leq L_{\omega'} \]

- assumptions regarding \((N, s)\): we list below all the conditions that must be satisfied in order for all the errors to be controlled in the process of proving these decay estimates in Section 3, 5, 6 and finding a finite bound of the mollified energy.

\[ (2.5) \quad N^{1-s}A^{\frac{p+1}{2}} \left( A^{-\frac{p+1}{2}} N^{s-1} \right)^{p\alpha} = o \left( (N^{s-1}A^{-\frac{p+1}{2}})^{\alpha} \right) \]

\[ (2.6) \quad s > \frac{3p-5}{2p} \]

\[ (2.7) \quad \frac{L^2(p-1)-(N^{1-s}A^{\frac{p+1}{2}})^{p-1}}{N^{\frac{s-1}{2}}} \ll 1 \]

\[ (2.8) \quad 2^C L_w^2(p-1) - \left( \max \left( \frac{L^2(p-1)-(N^{1-s}A^{\frac{p+1}{2}})^{p-1}}{N^{s-1}A^{\frac{p+1}{2}}}, \frac{1}{\beta N^{s-1}A^{\frac{p+1}{2}}} \right) \right)^p \ll 1 \]

\[ (2.9) \quad \frac{L^2(p-1)-(N^{1-s}A^{\frac{p+1}{2}})^{p-1}}{\beta N^{s-1}A^{\frac{p+1}{2}}} \ll 1 \]

\[ (2.10) \quad \frac{1}{\beta N^{s-1}A^{\frac{p+1}{2}}} \ll 1 \]

\[ (2.11) \quad \frac{(N^{1-s}A^{\frac{p+1}{2}})^{p-1}}{N^{\frac{s-1}{2}}} \ll 1 \]

and

\[ (2.12) \quad \frac{L^2(p-1)-(N^{1-s}A^{\frac{p+1}{2}})^{p+1}}{N^{\frac{s-1}{2}}} \ll (N^{s-1}A^{-\frac{p+1}{2}})^{\alpha} \]

, with

\[ (2.13) \quad \beta := \frac{\epsilon_0(N^{s-1})^{p+1}}{C_A N^{1-s}A^{\frac{p+1}{2}}} \]

\[ \eta := \eta_0(p) \text{ a small constant defined in Proposition 2.2 and } \epsilon_0 > 0 \text{ a fixed and small positive constant (say } \epsilon_0 = \frac{1}{1000}). \]
The first proposition shows that, on a subinterval \( J' \subset J \), if the mollified norm \( \|Iw\|_{L^2_tL^2_x(p-1)}(J') \) is small enough (in other words, we work locally in time), then one controls several norms on this subinterval. This proposition, combined with Proposition 1.5, 1.6 and 1.7 and the a priori assumptions (2.1) and (2.3), will allow to provide an a posteriori bound of the mollified energy and \( \|Iw\|_{L^2_tL^2_x(p-1)}(J') \) on an arbitrarily large time interval by an iteration process: see for example the proof of Theorem 1.1 and the proof of Proposition 2.4.

**Proposition 2.1. “Local Boundedness”** Let \( J' \subset J \). There exists \( \eta_0 > 0 \) such that if \( \eta \leq \eta_0 \) and \( \|Iw\|_{L^2_tL^2_x(p-1)}(J') \leq \eta \), then

\[
(2.14) \quad Z(J',w) \lesssim X_w
\]

The second proposition shows that if the mollified \( L^2_tL^2_x(p-1) \) norm of \( w \) on the interval \( J' \) is too large, then one can find a time \( T \in J' \) and a free Klein-Gordon solution \( v \) such that one can make the mollified energy of \( w' \) smaller than the sup of the mollified energy of \( w \) by a nontrivial amount. Furthermore, the mollified \( L^2_tL^2_x(p-1) \) norm of \( v \) on \((T,b)\) is small. The proof of Proposition 2.2 relies upon the combination of the I-method with a modification of arguments from Bourgain [1], or, more closely, Nakanishi [15, 16].

**Proposition 2.2. “Separation of the localized mollified energy if \( \|Iw\|_{L^2_tL^2_x(p-1)}(J') \) too large ”** Let \( M > 0 \). There exist \( \eta_1 := \eta_1(p) > 0, T \in J' \) and \( v \), solution of the free Klein-Gordon equation such that if

\[
(2.15) \quad \|Iw\|_{L^2_tL^2_x(p-1)}(J') \geq e^{C \max(M,e^{C(N^{1-s}A^{\frac{p-1}{2}})^{\alpha}})}
\]

then

\[
(2.16) \quad E(Iw',T) \leq \sup_{t \in J'} E(Iw) - (N^{s-1}A^{-\frac{p-1}{2}})^\alpha
\]

\[
(2.17) \quad E_v(Iv,T) \lesssim (N^{s-1}A^{-\frac{p-1}{2}})^\alpha
\]

and

\[
(2.18) \quad \|Iv\|_{L^2_tL^2_x(p-1)}(T,b) \lesssim \left(\frac{N^{1-s}A^{\frac{p+1}{2}}}{M^\alpha}\right)^\alpha
\]

The third proposition shows that if the mollified \( L^2_tL^2_x(p-1) \) norm of \( v \) is small enough then the mollified \( L^2_tL^2_x(p-1) \) norm of \( w \) can be estimated from the a priori upper bound (2.4) of the mollified \( L^2_tL^2_x(p-1) \) norm of \( w' \).

---

\(^6\)Recall that \( w' \) is the solution of (1.1) with the same data as \( w - v \) at time \( T \).
Proposition 2.3. “Perturbation argument”

Assume that

\begin{equation}
E_c(Iv) \leq (N^{s-1}A^{-\frac{p+1}{2}})^\alpha
\end{equation}

(2.19)

and

\begin{equation}
k \ll \frac{1}{C_2w'}
\end{equation}

(2.20)

Then

\begin{equation}
\|Iv\|_{L^2_{t}L^{2(p-1)}_{x}(Iv,0)} \leq k
\end{equation}

(2.21)

(2.22)

The last proposition shows that if the mollified energy is small enough at one time, we can control the mollified energy for all time along with the mollified \(L^2_{t}L^{2(p-1)}_{x}\) norm of the solution.

Proposition 2.4. “Mollified energy small at one time implies control of \(\|Iw\|_{L^2_{t}L^{2(p-1)}_{x}(R)}\) and control of \(\sup_{t \in R} E(Iw(t))\)” There exist \(0 < \tilde{\eta} \ll 1\) and \(\tilde{\eta} \in R\) such that if \(E(Iw(\tilde{t})) \leq \tilde{\eta}\) then

\begin{equation}
\|Iw\|_{L^2_{t}L^{2(p-1)}_{x}(R)} \lesssim \tilde{\eta}
\end{equation}

(2.23)

and

\begin{equation}
\sup_{t \in R} E(Iw(t)) \lesssim E(Iw(\tilde{t}))(1 + \beta)
\end{equation}

(2.24)

Remark 2.5. Ideally we would like to give more room to the variation of the mollified energy, i.e., \(\sup_{t} E(Iw(t)) \lesssim E(Iw(\tilde{t}))\) but we can’t. Indeed, the proof of Theorem 1.1 relies upon a modification of the Bourgain’s induction method on levels of energy. Using (2.16), we can construct a solution \(w'\) of (1.1) such that at one time, its mollified energy is smaller by a nontrivial amount only if we make the variation of \(E(Iw)\) small enough. It occurs that, by proving that \(\sup_{t} E(Iw(t)) \lesssim E(Iw(\tilde{t}))(1 + \beta)\), one can make the mollified energy decrease at the rate \((1 - \epsilon_0)(N^{s-1}A^{-\frac{p+1}{2}})^\alpha\) (see (2.30)). It is roughly equal to the one \(((N^{s-1}A^{-\frac{p+1}{2}})^\alpha)\) that we would have obtained if the variation were equal to zero.

We are now in position to prove Theorem 1.1. We define the following statement of induction for \(0 \leq l \leq \bar{l}\) with \(\bar{l} \sim (1 - \epsilon_0)^{-1}N^{2(1-s)}A^{p+1}(N^{s-1}A^{-\frac{p+1}{2}})^{-\alpha}\)

\(P(l)\): for \(i \geq l\) let \(C_i\) be the following set

\[C_i := \left\{ w_i : w_i \text{ solution of } (1.1), \exists t_i \in R \text{ s.t } E(Iw_i(t_i)) \leq C_iN^{2(1-s)}A^{p+1} - i(1 - \epsilon_0)(N^{s-1}A^{-\frac{p+1}{2}})^{\alpha} \right\}\]

\(\text{Recall that } N := N(A,p) \text{ and } s := s(A,p): \text{ therefore, they do not depend on } l\)
Then there exists \( \infty > L(i) \) such that
\[
\inf \{ \tilde{C}, w_i \in C_i \ and \ \|I w_i\|_{L^2_t L^{2(p-1)}_x (\mathbb{R})} \leq \tilde{C} \} = L(i)
\]
and
\[
\text{sup}_{t \in \mathbb{R}^+} E(I w_i(t)) \leq (C_E N^{2(1-s)} A^{p+1} - i(1 - \epsilon_0)(N^{s-1} \eta_1 A^{-\frac{p+1}{2}})^{\alpha})(1 + \beta)
\]
We shall prove by induction on \( l \) that \( \mathcal{P}(l) \) is true for all \( 0 \leq l \leq \tilde{l} \) and for \( (N, s) \) satisfying (2.5), ..., (2.12) with \( L_w, L_w' \) substituted for \( L(l), L(l + 1) \) respectively.

\( \mathcal{P}(l + 1) \Rightarrow \mathcal{P}(l) \). Indeed let \( w_i \in C_i \). We claim that (2.25) and (2.26) hold for \( i = l \) and some \( \infty > L(l) \geq 1 \) (to be determined shortly) if we are restricted to \([t_1, \infty)\). Indeed we introduce
\[
F := \left\{ T \in [t_1, \infty) : \text{sup}_{t \in [t, T]} E(I w_i(t)) \leq (C_E N^{2(1-s)} A^{p+1} - l(1 - \epsilon_0)(N^{s-1} \eta_1 A^{-\frac{p+1}{2}})^{\alpha})(1 + \beta) \right\}
\]
We claim that \( F = [t_1, \infty) \). Indeed
\( F \neq \emptyset \) since \( t_1 \in F \) by assumption.
- \( F \) is closed by continuity
- \( F \) is an open set of \([t_1, \infty)\). Indeed let \( \tilde{T} \in F \). Then by continuity, there exists \( \delta > 0 \) such that for all \( T \in (\tilde{T} - \delta, \tilde{T} + \delta) \cap [t_1, \infty) \) we have
\[
\sup_{t \in [t, T]} E(I w_i(t)) \leq Y_{w_i}(1 + 2\beta)
\]
with \( Y_{w_i} := C_E N^{2(1-s)} A^{p+1} - l(1 - \epsilon_0)(N^{s-1} \eta_1 A^{-\frac{p+1}{2}})^{\alpha} \) and
\[
\|I w_i\|_{L^2_t L^{2(p-1)}_x ([t_1, T])} \leq 2L(l)
\]
Now we divide \([t_1, T]\) into subintervals \((J_j)_{1 \leq j \leq m}\) such that \( \|I w_i\|_{L^2_t L^{2(p-1)}_x (J_j)} = \eta_0 \) and \( \|I w_i\|_{L^2_t L^{2(p-1)}_x (J_m)} \leq \eta_0 \). By (2.27), (2.28) and \( \mathcal{P}(l + 1) \) we can apply Proposition 2.1 on each \( J_j \) and get
\[
Z(w_i, J_j) \leq Y_{w_i}^{\frac{1}{2}}
\]
Now by applying Proposition 1.5 and iterating on \( j \) we see that in fact
\[
\sup_{t \in [t, T]} E(I w_i(t)) \leq (C_E N^{2(1-s)} A^{p+1} - l(1 - \epsilon_0)(N^{s-1} \eta_1 A^{-\frac{p+1}{2}})^{\alpha})(1 + \beta)
\]
, since \( (N, s) \) satisfies (2.9).

\(^8\)The important point is that the upper bound \( L(i) \) does not depend on the function \( w_i \). Since the most important variable is \( i \), we forget \( N, s \) and \( A \) in the proof, in order to avoid too much notation.

\(^9\)The reader can check that the estimates (2.25) and (2.26) hold if we are restricted to \([0, t_1]\) by slightly modifying the proof of (2.25) and (2.26), restricted to \([t_1, \infty)\).
Now we aim at controlling \( \|Iw_1\|_{L_t^2((p-1)^+)} \) and \( \|Iw_l\|_{L_t^2((p-1)^+)} \). To this end we let \( M \) be such that

\[(2.29) \quad \frac{(N^{1-s} \eta_1 A^{\frac{p+1}{2}})^\alpha}{M^\alpha} = \frac{1}{C^2 L^\alpha}[1]
\]

Let \( B > 0 \). If \( \|Iw_1\|_{L_t^2((p-1)^+)} \geq 2B \) then we can find \( \hat{t} \in [t_1, T] \) such that \( \|Iw_1\|_{L_t^2((p-1)^+)}(\hat{t}) \geq B \) and \( \|Iw_1\|_{L_t^2((p-1)^+)}(\hat{t}) \geq B \). Therefore if \( \mathcal{P}_l \) is such that \( \|Iw_1\|_{L_t^2((p-1)^+)}(\hat{t}) \) is true, we see that there exists \( T \in \hat{t} \) solution of (1.1) such that

\[(2.30) \quad B \leq Cc_{N^2(1-s)} A^{p+1} - (l + 1) \leq - (l - \alpha)(N^{1-s} \eta_1 A^{\frac{p+1}{2}})^\alpha
\]

, by (2.13). Therefore \( w_1 \in \mathcal{C}_{l+1} \). Applying \( \mathcal{P}_l+1 \) and Proposition 2.3 (possible, by (2.29)), we see that

\[(2.31) \quad B \leq \mathcal{L}(l + 1)
\]

and \( \|Iw_1\|_{L_t^2((p-1)^+)}(\hat{t}) \) is true. Therefore, we see that

\[(2.32) \quad L(l) \leq C_{\max} \left( C_{2C}^{\mathcal{L}(l+1)} N^{1-s} \eta_1 A^{\frac{p+1}{2}}, \epsilon C_{N^2(1-s) A^{\frac{p+1}{2}}} \right)^{C_{N^2(1-s) A^{\frac{p+1}{2}}}}
\]

and \( \mathcal{P}(l) \) holds.

Now assume that \( (N, s) \) satisfies the additional assumption

\[(2.33) \quad N^{1-s} = 2
\]

Iterating \( l \) times, we see that \( L(l) \leq X_{\bar{C}, \bar{s}} \) for all \( 0 \leq l \leq \bar{t} \). Therefore (1.7) holds.

Recall that \( (N, s) \) must satisfy (2.5),...,(2.12) with \( L_w, L_w' \) substituted for \( X_{\bar{C}, \bar{s}} \). Notice that (2.7), (2.8), (2.9) and (2.12) are the most constraining assumptions. Therefore if \( N \geq 2^{C_{X_{\bar{C}, \bar{s}}} A^C X_{\bar{C}, \bar{s}}} \), then all these assumptions are satisfied. Plugging these values of \( N \) into (2.33) we see that me must have \( 1 > s > \bar{s} \).
Global existence

We have just proved that

\[
\sup_{t \in \mathbb{R}} E(Iu(t)) \lesssim_A 1
\]

and

\[
\|Iu\|_{L^2_t L^{2(p-1)}_x} \lesssim_A 1
\]

By Plancherel we have \(\|\phi(T, \partial_t u(T))\|_{H^\infty H^{-1}} \lesssim E(Iu(T))\) for all \(T \in \mathbb{R}\). This proves global well-posedness of (1.1) with data \((u_0, u_1) \in H^s \times H^{s-1}\) such that \([u_0, u_1]\|_{H^s H^{s-1}} \leq A\) and \(1 > s > \delta := \delta(A, p)\).

Global estimates

Now we claim that

\[
Z_{m, s} (\mathbb{R}, u) \lesssim_A 1
\]

for all \(0 \leq m \leq s\). Let \(\mathcal{P} := (J_l = [a_l, b_l])_{1 \leq l \leq L}\) be a partition of \([0, \infty)\) such that \([Iu]\|_{L^2_t L^{2(p-1)}_x(J_l)} = \eta_l, \ l < L\) and \([Iu]\|_{L^2_t L^{2(p-1)}_x(J_L)} \leq \eta_0\), with \(\eta_0\) defined in Proposition 2.1. Notice that, by (2.35), this implies that \(L \lesssim_A 1\).

Moreover, by slightly modifying (3.2), a continuity argument and (2.34), we see that

\[
Z_{m, s} (J_l, u) \lesssim N^{1-s} A^{\frac{p+1}{p-1}} + Z_{m, s} (J_l, u) \left(\frac{\eta^p}{n^p} + \frac{Z^{p-1} (J_l, u)}{N^{\frac{p+1}{p-1}}}ight),
\]

\[
\lesssim N^{1-s} A^{\frac{p+1}{p-1}} + \eta_0 \lesssim_A 1
\], since \((N, s)\) satisfies (2.6) and (2.11). Therefore, iterating over \(l\), we get (2.36).

Scattering

Let \(v(t) := (u(t), \partial_t u(t)), \ v_0 := (u_0, u_1)\) and

\[
u_{nl}(t) = \begin{pmatrix}
\int_0^t \sin \left(\frac{(t-t')(D)}{\langle D \rangle}\right) \left(\left|u\right|^p (t') u(t')\right) \ dt' \\
\int_0^t \cos \left(\frac{(t-t')(D)}{\langle D \rangle}\right) \left(\left|u\right|^p (t') u(t')\right) \ dt'
\end{pmatrix}
\]

Then we get from (1.2) \(v(t) = K(t) v_0 - \nu_{nl}(t)\). Recall that the solution \(u\) scatters in \(H^p H^{p-1}\) if there exists \(v_{+0} := (u_{+0}, u_{+1})\) such that \(\|v(t) - K(t) v_{+0}\|_{H^p H^{p-1}} \to 0\) as \(t \to \infty\). In other words, since \(K\) is bounded on \(H^p H^{p-1}\), it suffices to prove that the quantity \(\|K^{-1}(t) v(t) - v_{+0}\|_{H^p H^{p-1}} \to 0\) as \(t \to \infty\). A computation shows that

\[
K^{-1}(t) = \begin{pmatrix}
\cos \left(\frac{t(D)}{\langle D \rangle}\right) & -\sin \left(\frac{t(D)}{\langle D \rangle}\right) \\
\langle D \rangle \sin \left(\frac{t(D)}{\langle D \rangle}\right) & \langle D \rangle \cos \left(\frac{t(D)}{\langle D \rangle}\right)
\end{pmatrix}
\]

But \(K^{-1}(t) v(t) = v_0 - K^{-1}(t) \nu_{nl}(t)\) and, by Proposition 1.3
\[ (2.37) \quad \|K^{-1}(t_1)u_{n_1}(t_1) - K^{-1}(t_2)u_{n_1}(t_2)\|_{H^s \times H^{s-1}} \lesssim \|u|^{p-1}u\|_{L_t^{\frac{q}{2}}L_x^{\frac{q}{2}}(\{(t_1, t_2)\)}} \]

But, plugging \((D)^{1-s}\) into (1.9) and modifying slightly (3.2), we get

\[ (2.38) \quad \|\langle D \rangle^{1-s}I(\|u|^{p-1}u)\|_{L_t^2L_x^{\frac{2}{p}}} \lesssim \|\langle D \rangle^{1-s}Iu\|_{L_t^2L_x^{\frac{2}{p}}(\{(t_1, t_2)\)}} \]

Therefore in view of (2.35), (2.36), (2.37) and (2.38), we see that the Cauchy criterion is satisfied by \(K^{-1}(t)v(t)\) and we conclude that \(K^{-1}(t)v(t)\) has a limit in \(H^s \times H^{s-1}\) as \(t\) goes to infinity. Moreover \(\lim_{t \to \infty} \|v(t) - K(t)v_+, 0\|_{H^s \times H^{s-1}} = 0\), with \(v_+, 0 := (u_+, 0, u_+ 1)\) given explicitly by

\[
\begin{align*}
&u_+, 0 := u_0 + \int_0^\infty \frac{\sin (\langle D \rangle^s t)}{\langle D \rangle^s t} u(t') \, dt' \\
u_+, 1 := u_1 - \int_0^\infty \cos (\langle D \rangle^s t) u(t') \, dt'.
\end{align*}
\]

3. PROOF OF PROPOSITION 2.1

In this section we prove Proposition 2.1. Plugging \(\langle D \rangle^{1-m}I\) into (1.9) we have

\[ (3.1) \quad Z_{m,s}(J', w) \lesssim X_w + \|\langle D \rangle^{1-m}I(\|u|^{p-1}u)\|_{L_t^{\frac{q}{2}}L_x^{\frac{2}{p}}(J')} \]

There are three cases:

- \(m \leq s\): we have

\[ (3.2) \quad Z_{m,s}(J', w) \lesssim X_w + \|\langle D \rangle^{1-m}Iu\|_{L_t^2L_x^{\frac{2}{p}}(J')} \|\|w|^{p-1}\|_{L_t^2L_x^{\frac{2}{p}}(J')} \]

\[ \lesssim X_w + Z_{m,s}(J', w) \|w|^{p-1}\|_{L_t^{\frac{2}{p}}L_x^{\frac{2}{p}}(J')} + \|P_{<N}w|^{p-1}\|_{L_t^{\frac{2}{p}}L_x^{\frac{2}{p}}(J')} \]

\[ \lesssim X_w + Z_{m,s}(J', w) \|\|w|^{p-1}\|_{L_t^{\frac{2}{p}}L_x^{\frac{2}{p}}(J')} + \|\|P_{<N}w|^{p-1}\|_{L_t^{\frac{2}{p}}L_x^{\frac{2}{p}}(J')} \]

\[ \lesssim X_w + Z_{m,s}(J', w) \left( \|w|^{p-1}\|_{L_t^{\frac{2}{p}}L_x^{\frac{2}{p}}(J')} + \frac{\|\langle D \rangle^{1-s}Iu\|_{L_t^{\frac{2}{p}}L_x^{\frac{2}{p}}(J')}}{N^{\frac{2}{2-p}}} \right). \]

Therefore, since \((N, s)\) satisfies (2.6) and (2.11), we see that, by a continuity argument, \(Z_{m,s}(J', w) \lesssim X_w\).

- \(m = 1\). We have
\[ Z_{m,s}(J, w) \lesssim X_w + \|I(|w|^{p-1}w)\|_{L_t^1L_x^2(J')} \]
\[ \lesssim X_w + \|P_{<N}w|^{p-1}P_{<N}w\|_{L_t^1L_x^2(J')} + \|P_{<N}w|^{p-1}P_{\geq N}w\|_{L_t^1L_x^2(J')} \]
\[ \lesssim X_w + \|P_{<N}w|^{p-1}P_{<N}w\|_{L_t^1L_x^2(J')} + \|P_{<N}w|^{p-1}P_{\geq N}w\|_{L_t^1L_x^2(J')} \]
\[ \lesssim X_w + B_1 + B_2 + B_3 + B_4. \]

But
\[ B_1 \lesssim \|Iw\|_{L_t^2(w)}, \]
\[ B_2 \lesssim \|Iw\|_{L_t^{6(p-1)}; L_x^2(J')} \|P_{\geq N}w\|_{L_t^\infty L_x^\infty(J')} \]
\[ B_3 \lesssim \frac{\|\langle D\rangle^{1-k}Iw\|_{L_t^{2(p-1)}; L_x^2(J')} \|Iw\|_{L_t^2 L_x^\infty(J')}}{N^{\frac{2-k}{2}}} \]
\[ B_4 \lesssim \frac{\|P_{\geq N}w\|_{L_t^{p} L_x^{2p}(J')}}{N^{\frac{2-k}{2}}} \]

and, by (3.2), (3.3), (3.4), (3.5), (3.6), (3.7) and a continuity argument we also get \( Z_{1,s}(J, w) \lesssim X_w \), since again, \((N, s)\) satisfies (2.11) and (2.6).

- \( s < m < 1 \): it follows by interpolating between \( m = s \) and \( m = 1 \).
4. Proof of Proposition 2.4

In this section we prove Proposition 2.4. The proof is made of two steps:

- **Control of** \( ||w||_{L^2_t L^2_x} \): by looking at the proof of Proposition 2.1, we realize that
  \[
  Z_{s_c, s}(\mathbb{R}, w) \leq \tilde{\eta} + \langle (D)^{1-s_c} w \rangle_{L^2_t L^2_x} + \frac{p-1}{p} \left( \frac{\beta}{N} \right) \left( \frac{\eta}{N} \right) \left( \frac{\eta}{N} \right)
  \]
  \[
  \leq \tilde{\eta} + \left( Z^p_{s_c, s}(\mathbb{R}, w) + Z^p_{s_c, s}(\mathbb{R}, w) \right)
  \]
  , where we used the Sobolev embedding \( ||w||_{L^2_t L^2_x} \leq \langle (D)^{1-s_c} w \rangle_{L^2_t L^2_x} \). Therefore, by a continuity argument, we see that \( Z_{s_c, s}(\mathbb{R}, w) \) if \( \tilde{\eta} < 1 \) and, consequently, (2.23) holds.

- **Control of** \( \sup_{t \in \mathbb{R}} E(Iw(t)) \).
  We define the set
  \[
  F := \left\{ T \in \mathbb{R}^+, \sup_{t \in [\bar{T}, \bar{T}+T]} E(Iw(t)) \leq E(Iw(\bar{T})) \left( 1 + \beta \right) \right\}
  \]
  - \( F \) is not empty since \( 0 \in F \).
  - \( F \) is closed by continuity.
  - \( F \) is open: given \( T \in F \), there exists by continuity \( \delta > 0 \) such that for \( T' \in (T, T+\delta) \), \( \sup_{t \in [\bar{T}-T', \bar{T}+T']} E(Iw(t)) \leq E(Iw(\bar{T})) \left( 1 + 2\beta \right) \). Using this inequality, the assumption \( E(Iw(\bar{T})) \leq \tilde{\eta} \) and (2.23), we see, by Proposition 2.1, that \( Z_{s_c, s}([\bar{T} - T', \bar{T} + T'], w) \leq E^{4}(Iw(t)) \) for all \( m \in [0, 1] \), if \( \tilde{\eta} < 1 \). Combining these bounds with Proposition 1.5, we see that in fact (2.24) holds, since \( (N, s) \) satisfies (2.10).

Therefore \( F = \mathbb{R}^+ \).

5. Proof of Proposition 2.2

In this section we prove Proposition 2.2. The proof of Proposition 2.2 relies upon three lemmas that we show in the next subsections. Throughout the proof of these lemmas \( \eta \) denotes a positive constant such that \( \eta \leq \eta_0 \), with \( \eta_0 \) defined in Proposition 2.1.

The first lemma shows that if the mollified \( L^2_t L^2_x \) norm of the solution on a subinterval \( J' \subset J \) is substantial, then this also means that the potential term of the mollified energy and the size of this subinterval are substantial.

**Lemma 5.1.** “**Concentration of** \( ||w||_{L^2_t L^2_x} \) **implies concentration of potential term of the mollified energy**” Assume that \( ||w||_{L^2_t L^2_x} = \eta \). Then there exist a subinterval \( \bar{K}' \subset J' \), a number \( R := C(\eta^{-1} A^{\frac{\beta}{2}} N^{-s}) > 0 \) and a point \( \bar{x} \in \mathbb{R}^3 \) such that
for all $\log$ growth of size of intervals on which there is

\begin{equation}
|K'| = C^{-1}(\eta A^{-\frac{p+1}{2p}} N^{s-1})^\alpha
\end{equation}

for all $t \in K'$

\begin{equation}
\int_{|x-x'| \leq R} |w(t,x)|^{p+1} \, dx \gtrsim (A^{-\frac{p+1}{2p}} N^{s-1})^\alpha
\end{equation}

The second lemma shows that if we consider a partition of an interval into subintervals where the mollified $L^2_x L^p_t$ norm of the solution concentrates, then these subintervals must be large on average. In order to prove this lemma, we shall mostly use the previous lemma and the Almost Morawetz-Strauss estimate (1.13).

**Lemma 5.2.** "Logarithmic growth of size of intervals on which there is concentration"

Let $(J'_j = [a'_j, a'_{j+1}])_{1 \leq j \leq J}$ be a partition of $J'$ such that $\|w\|_{L^2_x L^p_t(J'_j)} = \eta$, $j < j'$, and $\|w\|_{L^2_x L^p_t(J'_j)} \lesssim \eta$. Then there exists $t'_j \in J'_j$ such that

\begin{equation}
\sum_{j=1}^{J_j' \leq J + 1} \frac{1}{t'_j} \lesssim (N^{1-s} A^{\frac{1}{p+1}})^\alpha.
\end{equation}

The third lemma shows that if the mollified norm of the solution is too large then we can find a large subinterval where some Strichartz norms (and other norms) are small compared with the concentration.

**Lemma 5.3.** "If $\|w\|_{L^2_x L^{p+1}_t(J')} \gtrsim R$ large enough then there exists a large subinterval s.t some Strichartz norms are smaller than the concentration of mollified energy"

Let $M > 0$. Let $J' \subset J'$ and let

\[
\begin{aligned}
\tilde{Z}(J', w) &= \|D_1D^{-\frac{1}{2}}Iw\|_{L^4_x L^2_t(J')} + \|w\|_{L^2_x L^p_t(J')} \\
&+ \|D\|^{-\frac{1}{2}} Iw\|_{L^{\frac{2p}{p+1}}_x L^{\frac{4p}{p+1}}_t(J')} + \|\langle D \rangle\|^{-\frac{1}{2}} Iw\|_{L^{\frac{4p}{p+1}}_x L^{4}_t(J')} \\
&+ \|\langle D \rangle\|^{-\frac{1}{2}} Iw\|_{L^{4}_x L^{4}_t(J')} + \|\langle D \rangle\|^{-\frac{1}{2}} Iw\|_{L^{4}_x L^{\frac{4p}{p-1}}_t(J')}
\end{aligned}
\]

Then there exist $\eta_1 := \eta_1(p) << 1$, $R' \in (1, \infty)$, $\bar{x} \in \mathbb{R}^3$ and $[S, T] \subset J'$ such that for all $\epsilon << 1$, if

\begin{equation}
\|w\|_{L^2_x L^{p+1}_t(J')} \gtrsim C^{\max (C^{1-s} A^{\frac{1}{p+1}} N^{s-1})^\alpha}
\end{equation}

then

\begin{equation}
\tilde{Z}([S, T], w) \lesssim 3(\eta A^{-\frac{p+1}{2p}} N^{s-1})^\alpha \leq E(Iw(S), B(\bar{x}, R'))
\end{equation}

and

\begin{equation}
|T - S| \geq MR'
\end{equation}
and

(5.7) \[ \left\| \frac{I_w(S)}{x-x'} \right\|_{L^2} \leq \varepsilon. \]

Assuming that Lemma 5.1, Lemma 5.2 and Lemma 5.3 are true, let us prove Proposition 2.2. We apply Lemma 5.3 with \( \epsilon := \frac{1}{10}(N^{s-1} \eta_1 A^{-\frac{p+1}{2}})^{\alpha} N^{s-1} A^{-\frac{p+1}{2}} \) 10. The proof is made of several steps:

- **Construction of the free Klein-Gordon equation** \( v \)

  Let \( P(y) := \{ y \in \mathbb{R}^3, E(Iw(S), B(y, 1)) \leq \frac{1}{10}(N^{s-1} \eta_1 A^{-\frac{p+1}{2}})^{\alpha} \} \). Then by (2.1) there exists \( \bar{x} \in \mathbb{R}^3 \) such that \( |\bar{x} - \bar{x}| \leq \frac{1}{2} \). Then by this observation, notice (see [15]) that, by continuity, there exists \( \Gamma \in [\frac{1}{2}, R'+ C(N^{1-s} \eta_1 A^{-\frac{p+1}{2}})^{\alpha}] \) such that

\[ (5.8) \quad E(Iw(S), B(\bar{x}, \Gamma)) = 3(N^{s-1} \eta_1 A^{-\frac{p+1}{2}})^{\alpha}, \]

since

\[ E(Iw(S), B(\bar{x}, 0.5)) \leq E(Iw(S), B(\bar{x}, 1)) \leq \frac{1}{10}(N^{s-1} \eta_1 A^{-\frac{p+1}{2}})^{\alpha} \]

and

\[ E(Iw(S), B(\bar{x}, R')) \geq 3(N^{s-1} \eta_1 A^{-\frac{p+1}{2}})^{\alpha} \]

Let \( v \) be the solution of the free Klein-Gordon equation with data

\[ \begin{aligned}
  v(S) &:= I^{-1}(\chi \left( \frac{x-x'}{R} \right) Iw(S)) \\
  \partial_t v(S) &:= I^{-1}(\chi \left( \frac{x-x'}{R} \right) \partial_t Iw(S))
\end{aligned} \]

, where \( \chi \) is a smooth function such that \( \chi(x) = 1 \) if \( |x| \leq \frac{1}{2} \) and \( \chi(x) = 0 \) if \( |x| \geq 1 \). A computation shows that

(5.9) \[ E(v, S) \leq \left| E(Iw(S), B(\bar{x}, \Gamma)) \right| + O \left( \int_{|x| \leq 2R} |Iw(S)|^2 \frac{dx}{R} \right) + O \left( \int_{|x| \leq 2R} |\nabla Iw(S)|^2 \frac{dx}{R} \right) \]

\[ \leq (N^{s-1} \eta_1 A^{-\frac{p+1}{2}})^{\alpha} + O \left( N^{1-s} A^{\frac{p+1}{2}} \frac{Iw(S)}{|x-x'|} \right) \right\|_{L^2} \] + O \left( \left\| \frac{Iw(S)}{|x-x'|} \right\|_{L^2} \right)

\[ \lesssim (N^{s-1} \eta_1 A^{-\frac{p+1}{2}})^{\alpha} \]

and

\[ \eta \text{ is not randomly chosen: see (5.9) and (5.10)} \]
(5.10) 
\[ E(Iw - Iv, S) \leq \frac{1}{2} \int_{\mathbb{R}^3} (1 - \chi^2 (\frac{|x|}{\eta})) |\nabla Iw(S)|^2 dx + \frac{1}{p+1} \int_{\mathbb{R}^3} (1 - \chi^{p+1} (\frac{|x|}{\eta})) |Iw(S)|^{p+1} dx \]
\[ + \frac{1}{2} \int_{\mathbb{R}^3} (1 - \chi^2 (\frac{|x|}{\eta})) |\partial_t Iw(S)|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} (1 - \chi^2 (\frac{|x|}{\eta})) |Iw(S)|^2 dx \]
\[ + O \left( \frac{1}{T} \int_{|x-t| \leq 2T} |Iw(S)| |\nabla Iw(S)| dx + \frac{1}{T} \int_{|x-t| \leq 2T} |Iw(S)|^2 dx \right) \]
\[ \leq \sup_{t \in [T]} E(Iw(t)) - 3(N^{-1} \eta A^{-\frac{p+1}{2}})^\alpha + O \left( N^{-1} A \frac{p+1}{2} \parallel Iw(S) \parallel_{L^2} \right) \]
\[ + O \left( \parallel Iw(S) \parallel_{(\frac{2-p}{2})} \right)^2 \]
\[ \leq \sup_{t \in [T]} E(Iw(t)) - 2(N^{-1} \eta A^{-\frac{p+1}{2}})^\alpha , \]

• Proof of the decay (2.18)
By Sobolev, interpolation, (5.6) and the following dispersive estimate (see [9], Lemma 2.1)
\[ \|e^{it(D)^{\frac{\alpha}{\gamma}}} \|_{B^\frac{\gamma}{2}} \lesssim \frac{1}{|t|^\frac{\gamma}{2}} \| \phi \|_{B^\gamma} \]
we have, by letting \( \gamma := 1 - \frac{4}{(p-1)(3-2v)} \),
\[ \|Iv\|_{L^2(T)} \lesssim \| (D)^{-\frac{1}{2}} Iv \|_{L^\infty} \| (D)^{-\frac{1}{2}} Iv \|_{L^2(T)} \]
\[ \lesssim (N^{-1} \eta A^{-\frac{p+1}{2}})^\alpha \| Iv \|_{L^\infty} \| Iv \|_{L^2(T)} \]
\[ \lesssim (N^{-1} \eta A^{-\frac{p+1}{2}})^\alpha \frac{1}{|T-S|^{\frac{\gamma}{2}}} \left( \| Iv(S) \|_{L^2} + \| \partial_t Iv(S) \|_{L^2} \right) \]
\[ \lesssim (N^{-1} \eta A^{-\frac{p+1}{2}})^\alpha \frac{1}{|T-S|^{\frac{\gamma}{2}}} \left( \| \nabla Iv(S) \|_{L^2} + \| \partial_t Iv(S) \|_{L^2} \right) \]
\[ \lesssim (N^{-1} \eta A^{-\frac{p+1}{2}})^\alpha \frac{1}{|T-S|^{\frac{\gamma}{2}}} \left( \| \nabla Iv(S) \|_{L^2} + \Gamma \parallel \nabla Iw(S) \parallel_{L^2} + \Gamma \parallel \partial_t Iw(S) \parallel_{L^2} \right) \]
\[ \lesssim (N^{-1} \eta A^{-\frac{p+1}{2}})^\alpha , \]
Therefore (2.18) follows.

• Proof of the separation of the localized mollified energy (2.16).
Notice first, that, in view of (1.9) and (5.9), we have
\[ (5.12) \]
\[ Z(v, \mathbb{R}) \lesssim (N^{-1} \eta A^{-\frac{p+1}{2}})^\alpha \]
and therefore, in view of (5.5), we have
\[ (5.13) \]
\[ \tilde{Z}(w', [S, T]) \lesssim (N^{-1} \eta A^{-\frac{p+1}{2}})^\alpha \]
We compute
\[ \partial_t E(Iw', t) = \int_{\mathbb{R}^3} \Re \left( \partial_t Iw' \left( \partial_t Iw' - \Delta Iw' + Iw' + F(Iw') \right) \right) dx \]
\[ = \int_{\mathbb{R}^3} \Re \left( \partial_t Iw' \left( F(Iw') - 1F(w') \right) \right) dx . \]
Therefore \( E(Iw', T) - E(Iw', S) = X_{1,1} + X_{1,2} + X_2 \) with
\[ X_{1,1} := \int_S^T \int_{\mathbb{R}^3} \Re \left( \partial_t Iw' (F(w) - IF(w)) \right) \, dx \, dt \]
\[ X_{1,2} := \int_S^T \int_{\mathbb{R}^3} \Re \left( \partial_t Iw' (F(Iw) - F(w)) \right) \, dx \, dt \]
\[ X_2 := \int_S^T \int_{\mathbb{R}^3} \Re \left( \partial_t Iw' (F(Iw') - F(Iw)) \right) \, dx \, dt \]

It is enough to estimate \( X_{1,1}, X_{1,2} \) and \( X_2 \). By the fundamental theorem of calculus we have |\( F(w) - F(Iw) \)\| \( \lesssim \max(|Iw|^{p-1},|w|^{p-1})|Iw - w| \) and therefore

\[
X_{1,2} \lesssim \| \partial_t Iw' \|_{L_t^p L_x^2(S,T)} \times \left( \| P_{<N} w \|_{L_t^{4(p-1)} L_x^{4(p-1)}(S,T)} \frac{4(p-1)}{L_t^{p-1} L_x^{p-1}(S,T)} \right) \left( \| P_{\geq N} w \|_{L_t^p L_x^p(S,T)} \right) \left( \| (D)^{1-\frac{q}{p}} Iw \|_{L_t^p L_x^p(S,T)} \right)
\]

\[
\lesssim \frac{1}{N^{1-s}} \| \partial_t Iw' \|_{L_t^p L_x^2(S,T)} \times \left( \| (D)^{1-\frac{q}{p}} Iw \|_{L_t^p L_x^p(S,T)} \right) \left( \| P_{\geq N} w \|_{L_t^p L_x^p(S,T)} \right) \left( \| P_{\geq N} w \|_{L_t^{p-1} L_x^{p-1}(S,T)} \right)
\]

\[
\lesssim N^{1-s} A^{\frac{q}{p}} \left( A^{-\frac{q}{p} + \eta} N^{-s+1} \right) \]

\[
= o \left( (N^{s-1} \eta A^{-\frac{q}{p} + \eta})^\alpha \right),
\]

since \((N,s)\) satisfies (2.5).

We turn to \( X_{1,1} \). We use an argument in [20]: on low frequencies we use the smoothness of \( F \) (\( F \) is \( C^1 \)) and on high frequencies, we use the regularity of \( u \) (in \( H^s \)). We have, by the fundamental theorem of calculus

\[
F(w) := F \left( P_{<N} w + P_{\geq N} w \right)
\]
\[
= F \left( P_{<N} w \right) + \int_0^1 \left[ P_{<N} w + y P_{\geq N} w \right]^{p-1} \, dy \left( P_{\geq N} w \right)
\]
\[
+ \left( \int_0^1 P_{<N} w + y P_{\geq N} w \right) \left( P_{\geq N} w \right) \left( P_{\geq N} w \right).
\]

Therefore

\[
(5.15) \qquad X_{1,1} \lesssim \| \partial_t Iw' \|_{L_t^p L_x^2(\mathbb{R}^3)} (X_{1,1,1} + X_{1,1,2} + X_{1,1,3})
\]
\[
\lesssim N^{1-s} A^{\frac{q}{p}} (X_{1,1,1} + X_{1,1,2} + X_{1,1,3})
\]

with

\[
X_{1,1,1} := \| P_{\geq N} F(P_{<N} w) \|_{L_t^p L_x^2(\mathbb{R}^3)}
\]
\[
X_{1,1,2} := \| P_{<N} w \|^{p-1} P_{\geq N} w \|_{L_t^p L_x^2(\mathbb{R}^3)}
\]
\[
X_{1,1,3} := \| P_{\geq N} w \|_p \left( P_{\geq N} w \right)
\]

We have
\[ X_{1,1,1} \lesssim \frac{1}{N} \| \nabla P_{\mu N \nu} \|_{L^2(S,T)} \]
\[ \lesssim \frac{1}{N} \| P_{\mu N \nu} \|_{L^2(S,T)} \left\| \nabla P_{\mu N \nu} \right\|_{L^2(S,T)} \left\| \frac{4}{(p-1)} \frac{4(p-1)}{L^2_{(S,T)}} \frac{4(p-1)}{L^2_{(S,T)}} \left\| (D)^{-1} I w \right\|_{L^2_{(S,T)}} \right\|_{L^2_{(S,T)}} \]
\[ \lesssim \frac{1}{N} \| P_{\mu N \nu} \|_{L^2(S,T)} \left\| (D)^{-1} I w \right\|_{L^2_{(S,T)}} \left\| \frac{4}{(p-1)} \frac{4(p-1)}{L^2_{(S,T)}} \frac{4(p-1)}{L^2_{(S,T)}} \left\| (D)^{-1} I w \right\|_{L^2_{(S,T)}} \right\|_{L^2_{(S,T)}} \]
\[ \lesssim \langle N^{-\frac{1}{2}} \frac{1}{L^2_{(S,T)}} \rangle^p , \]

\[ X_{1,1,2} \lesssim \| P_{\mu N \nu} \|^p_{L^2_{(S,T)}} \left\| (D)^{-1} I w \right\|_{L^2_{(S,T)}} \left\| (D)^{-1} I w \right\|_{L^2_{(S,T)}} \]
\[ \lesssim \langle N^{-\frac{1}{2}} \frac{1}{L^2_{(S,T)}} \rangle^p , \]

and

\[ X_{1,1,3} \lesssim \| P_{\mu N \nu} \|^p_{L^2_{(S,T)}} \left\| \frac{4}{(p-1)} \frac{4(p-1)}{L^2_{(S,T)}} \frac{4(p-1)}{L^2_{(S,T)}} \left\| (D)^{-1} I w \right\|_{L^2_{(S,T)}} \right\|_{L^2_{(S,T)}} \]
\[ \lesssim \langle N^{-\frac{1}{2}} \frac{1}{L^2_{(S,T)}} \rangle^p , \]

Therefore \( X_{1,1} = o \left( \left( N^{s-1} \eta \nu A^{-\frac{4}{n+1}} \right)^\alpha \right) \) since, again, \((N,s)\) satisfies (2.5).

Now we turn to \( X_2 \). By (5.13) and (5.5), we have

\[ (5.16) \]
\[ X_2 \lesssim \| \partial_t (D)^{-\frac{1}{2}} I w \|_{L^4_{(S,T)}} \left\| (D)^{-\frac{1}{2}} (F(I w) - F(I w)) \right\|_{L^2_{(S,T)}} \]
\[ \lesssim \| \partial_t (D)^{-\frac{1}{2}} I w \|_{L^4_{(S,T)}} \left\| F(I w) \right\|_{L^2_{(S,T)}} \left\| (D)^{-\frac{1}{2}} I w \right\|_{L^2_{(S,T)}} \]
\[ \lesssim \langle (N^{-\frac{1}{2}} \frac{1}{L^2_{(S,T)}}) \rangle^p , \]

Therefore (2.16) holds since, again, \((N,s)\) satisfies (2.5). Notice also that, in view of (5.12), we have (2.17).

5.1. **Proof of Lemma 5.1.** The proof is made of five steps:

- **Control of auxiliary norms.** Let \( \hat{Z}(J) \) be such that

\[ \hat{Z}(J) = (\langle D \rangle^{1-\frac{1}{p}} I w \| L^2_{(S,T)}(J') + \langle D \rangle^{1-\frac{1}{p}} I w \| L^2_{(S,T)}(J') \)
\[ \lesssim \left\{ \begin{array}{ll}
\langle (D)^{1-\frac{1}{p}} I w \| L^2_{(S,T)}(J') \rangle, & \text{if } p \geq 4 \\
\langle (D)^{1-\frac{1}{p}} I w \| L^2_{(S,T)}(J') \rangle, & \text{if } p < 4
\end{array} \right. \]

(5.17)

By (2.1) we have
\( \hat{Z}(\bar{J}) \lesssim N^{1-s} A^{\frac{p+1}{2}} + \| (D)^{1-\frac{1}{2}} I (|w|^{p-1} w) \|_{L^2_t L^4_x(\bar{J})} \)

\[ \lesssim N^{1-s} A^{\frac{p+1}{2}} + \| (D)^{1-\frac{1}{2}} I w \|_{L^2_t L^4_x(\bar{J})} \left( \| \mathcal{P} \ll N w \|_{L^2_t L^4_x(\bar{J})}^{p-1} + \| P N w \|_{L^2_t L^4_x(\bar{J})} \right) \]

\[ \lesssim N^{1-s} A^{\frac{p+1}{2}} + \| w \|_{L^2_t L^4_x(\bar{J})} \left( \| (D)^{1-\frac{1}{2}} I w \|_{L^2_t L^4_x(\bar{J})} \right) \]

Applying a continuity argument, we see that

\[ (5.18) \quad \hat{Z}(\bar{J}) \lesssim N^{1-s} A^{\frac{p+1}{2}}, \]

since \((N, s)\) satisfies (2.6) and (2.11).

- **Lower bound of the size of \( \bar{J} \).**

  If \( p \geq 4 \) then, by (5.18) we have \(^{11}\)

  \[ \eta = \| w \|_{L^2_t L^4_x(\bar{J})} \]

  \[ \lesssim |\bar{J}| \left\{ \begin{array}{l}
  N^{1-s} \quad \text{if } p = 4, \\
  N^{1-s} A^{\frac{p+1}{2}} \quad \text{if } p > 4,
  \end{array} \right. \]

  and if \( p < 4 \)

  \[ \eta = \| w \|_{L^2_t L^4_x(\bar{J})} \]

  \[ \lesssim |\bar{J}| \left\{ \begin{array}{l}
  N^{1-s} \quad \text{if } p = 4, \\
  N^{1-s} A^{\frac{p+1}{2}} \quad \text{if } p > 4.
  \end{array} \right. \]

  Therefore

\[ (5.19) \quad |\bar{J}| \gtrsim (A^{-\frac{p+1}{2}} N^{s-1})^\alpha \]

- **Lower bound of \( \| P_M w \|_{L^\infty_t L^\infty_x(\bar{J})} \) for some \( M \in 2^N \) such that \( M \leq f(A, N) \)**

  We have, by (5.18)

  \[ \eta = \| w \|_{L^2_t L^4_x(\bar{J})} \]

  \[ \leq \| (D)^{1-\frac{1}{2}} I w \|_{L^2_t L^4_x(\bar{J})} \| (D) \times (\frac{1}{2} + N) I w \|_{L^\infty_t L^\infty_x(\bar{J})} \]

  \[ \lesssim A^{\frac{p+1}{2}} N^{\frac{2(1-s)}{p+1}} \| (D) \times (\frac{1}{2} + N) I w \|_{L^\infty_t L^\infty_x(\bar{J})} \]

  Therefore \( \| (D) \times (\frac{1}{2} + N) I w \|_{L^\infty_t L^\infty_x(\bar{J})} \gtrsim (A^{-\frac{p+1}{2}} N^{s-1})^\alpha \) and, by the pigeon-hole principle, this implies that there exists \( M \in 2^N \) such that

\(^{11}\)The sign \( \pm \) denotes the plus or minus sign. It is not important to know whether it is a plus or minus sign: see the next computations. Therefore we have decided not to determine the sign.
(5.20) \[ \|P_M w\|_{L^\infty_t L^2_x(J')} \gtrsim (M)^{\frac{1}{3+\alpha}} (A^{-\frac{2+\alpha}{1+\alpha}} N^{s-1})^\alpha. \]
Moreover
(5.21) \[ \|P_M w\|_{L^\infty_t L^2_x(J')} \lesssim (M)^{\frac{1}{2}} \|\partial_t w\|_{L^2_t L^2_x} \lesssim (M)^{\frac{1}{2}} N^{1-s} A^\frac{2\alpha}{1+\alpha}. \]
Therefore, combining (5.20) and (5.21), we see that
(5.22) \[ (M) \lesssim (A^{-\frac{2+\alpha}{1+\alpha}} N^{1-s})^\alpha. \]

- Control of \(|P_M w(t, \tilde{x}')|\) for some \(\tilde{x}' \in \mathbb{R}^3\) and for all \(t \in \tilde{K}', \tilde{K}' \subset J'\) to be defined shortly. By (5.20), there exists \((\tilde{t}', \tilde{x}')\) such that
(5.23) \[ |P_M w(\tilde{t}', \tilde{x}')| \gtrsim (M)^{\frac{1}{3+\alpha}} (A^{-\frac{2+\alpha}{1+\alpha}} N^{s-1})^\alpha. \]
But, by the fundamental theorem of calculus and (2.1), we see that
(5.24) \[ |P_M w(t, \tilde{x}') - P_M w(\tilde{t}', \tilde{x}')| \lesssim \sup_{s \in [\tilde{t}', t]} \|\partial_s w(s)\|_{L^2_x} (M)^{\frac{1}{2}} |t - \tilde{t}'| \lesssim (N^{1-s} A^\frac{2\alpha}{1+\alpha})^\alpha |t - \tilde{t}'|, \]
Therefore, in view of (5.19), (5.23) and (5.24), we have
(5.25) \[ |P_M w(t, \tilde{x}')| \gtrsim (\eta A^{-\frac{2+\alpha}{1+\alpha}} N^{s-1})^\alpha \]
for all \(t \in [\tilde{t}', \tilde{t}' + \delta] \subset J'\) or \(t \in [\tilde{t}' - \delta, \tilde{t}'] \subset J'\) with \(\delta := C^{-1}(A^{-\frac{2+\alpha}{1+\alpha}} N^{s-1})^\alpha\).

- Lower bound of potential mollified energy. Let \(R > 0\) to be fixed shortly. Let \(\Psi := \psi\) if \(M > 1\) and \(\Psi := \phi\) is \(M = 1\). By (5.25) we have
\[ M^3 (B_1 + B_2) \gtrsim (\eta A^{-\frac{2+\alpha}{1+\alpha}} N^{s-1})^\alpha \]
with \(B_1 := \int_{|w| \leq R} |\tilde{\Psi}(M y)| |w(t, \tilde{x}' - y)| dy\) and \(B_2 := \int_{|w| \geq R} |\tilde{\Psi}(M y)| |w(t, \tilde{x}' - y)| dy\). We have
\[ B_1 \lesssim (\int_{|w| \leq R} |\tilde{\Psi}(M y)|^{\frac{p+1}{p}} dy)^{\frac{p}{p+1}} (\int_{|w| \leq R} |w(t, \tilde{x}' - y)|^{p+1} dy)^{\frac{1}{p+1}} \lesssim (\int_{|w - x'| \leq R} |w(t, \tilde{x}' - y)|^{p+1} dy)^{\frac{1}{p+1}} M^{-\frac{3p}{p+1}} \]
and
\[ B_2 \lesssim M^{-\frac{3}{2}} \|\tilde{\Psi}\|_{L^2(|w| \geq M R)} \|w(t)\|_{L^2_x} \lesssim M^{-\frac{3}{2}} \|\tilde{\Psi}\|_{L^2(|w| \geq M R)} N^{1-s} A^\frac{2\alpha}{1+\alpha}. \]
Therefore, by the inequality \(\|\tilde{\Psi}\|_{L^2(|w| \geq M R)} \lesssim \frac{1}{(MR)^{\frac{3}{2}}}\), we see that if
\[ ^{12}\text{Here } f \text{ denotes the inverse Fourier transform of a function } f. \]
We construct (see \cite{15}) a set of weighted long time estimate and lower bound of mollified energy, restricted to a ball.

\[ R := C(A^{-\frac{p+1}{2}} N^{1-s})^\alpha \]
then \(M^{3-\frac{3}{2}}\|\Psi\|^2_{L^2_{|x| \geq MR}} N^{1-s} A^{-\frac{p+1}{2}} < \theta (A^{-\frac{p+1}{2}} N^{s-1})^\alpha\) and therefore

\[ \int_{|y - x'| \leq R} |Iw(t, y)|^{p+1} \, dy \lesssim (A^{-\frac{p+1}{2}} N^{s-1})^\alpha \]
for all \(t \in K' \).

5.2. Proof of Lemma 5.2. Recall that, by Lemma 5.1, on each \(J'_j\), there exist \(x'_j \in \mathbb{R}^3\) and \(K'_j = [t'_j, t'_{j+1}] \subset J'_j\) such that

\[ \int_{|x-x'| \leq R} |Iw(t, x)|^{p+1} \, dx \lesssim (A^{-\frac{p+1}{2}} N^{s-1})^\alpha \]
for all \(t \in K'_j\), with \(R = C(A^{-\frac{p+1}{2}} N^{1-s})^\alpha\) and

\[ |K'_j| = C^{-1}(A^{-\frac{p+1}{2}} N^{s-1})^\alpha \]
The proof is made of three steps:

- We construct (see \cite{15}) a set \(\mathcal{P} := \{j_1, \ldots, j\} \subset [1, \ldots, \tilde{j}\} \]. Initially \(j_1 = 1\).

  Then let \(j_{k+1}\) be the minimal \(j \in [1, \tilde{j}]\) such that

\[ |x'_j - x'_{j_{k+1}}| \geq |t'_j - t'_{j_{k+1}}| + 100R \]

for \(j := j_1, \ldots, j_k\). Observe that \(J' = (J'_j)_{1 \leq j \leq \tilde{j}} = \bigcup_{j_k \in \mathcal{P}} A'_{j_k}\) with \(A'_{j_k} := \{j'_t, j'_s \geq l \geq j_k, \text{ and } |x'_{j'_t} - x'_t| < |t'_s - t'_t| + 100R\}\).

- Weighted long time estimate and lower bound of mollified energy, restricted to a ball.

  We chop \(J'\) into subintervals \((\tilde{J}'_q)_{1 \leq q \leq Q}\) such that \(\|Iw\|_{L^2_{t=1-s} - L^2_{t=1-s} + \tilde{j}} = \eta_0, \|Iw\|_{L^2_{t=1-s} - L^2_{t=1-s} + \tilde{j}} \leq \eta_0\). By (2.3) we have

\[ Q \lesssim L^2_{t=1-s} \]

and combining this bound with Proposition 2.1, Proposition 1.7, Proposition 1.6 and (5.29) we see that, by iteration

\[ \int J \int R^2 |Iw(t, x)|^{p+1} \frac{\|dx \, dt}{|x-x'|} \lesssim N^{2(1-s)} A^{p+1} \],

since \((N, s)\) satisfies (2.7). Therefore

\[ \int J \int R^2 |Iw(t, x)|^{p+1} \frac{\|dx \, dt}{|x-x'|} \lesssim N^{2(1-s)} A^{p+1}. \]

This proves the weighted long-time estimate. Next, let us prove the following result:

Result: Let \(x'_a \in \mathbb{R}^3\) and let \(t'_b \geq t'_a\). Then

\[ E \left( Iw(t'_b), B(x'_a, R + t'_b - t'_a) \right) \geq E \left( Iw(t'_a), B(x'_a, R) \right) + o \left( (A^{-\frac{p+1}{2}} N^{s-1})^\alpha \right). \]
Indeed, by integrating the identity
\[
\partial_t \left( \frac{1}{2} |\partial_t w|^2 + \frac{1}{2} \nabla |w|^2 + \frac{|w|^{p+1}}{p+1} - \frac{|w|^2}{2} \right) - \partial_x \left( (\Re(\partial_t \overline{w} \partial_x w)) \right) + R(\partial_t w ) = 0
\]
inside the truncated cone \( M := \{(t,x), t \in (t'_a,t'_b), t = t'_a - R \geq |x - x'_a| \}\)
we have
\[
(5.33)
E \left( I w(t'_a), B(x'_a, R + t'_b - t'_a) \right) - E \left( I w(t'_a), B(x'_a, R) \right) = \frac{1}{\sqrt{2}} \int_{\partial M} |w|^2 + \frac{|w|^{p+1}}{p+1} \, d\sigma
+ \frac{1}{\sqrt{2}} \int_{\partial M} \left| \frac{x-x'_a}{|x-x'_a|} \partial_t w + \nabla w \right|^2 \, d\sigma - \int_{\partial M} R(\partial_t w (I F(w) - F(I w))) \, dx dt.
\]
Therefore, by repeating the steps from (5.29) to (5.30) and using (1.11) instead of Proposition 1.6, we get after iteration, (5.32), since \((N,s)\) satisfies (2.12). Consequently, by our choice of \(R, (5.2), (5.27), (5.26)\) and (5.31) we have
\[
N^{2(1-s)A^{p+1}} \text{card} \mathcal{P} \gtrsim \sum_{j_k \in \mathcal{P}} \int_{J'k} \int_{\mathbb{R}^3} \frac{|I w(t,x)|^{p+1}}{|x-x_{jk}|} \, dx dt
\]
\[
\gtrsim (N^{s-1}A^{-\frac{p+1}{4}})^{\alpha} \sum_{j_k \in \mathcal{P}} \int_{J'k} \int_{J'k} |x-x'_{jk}|^{-|t-t'_{jk}|} \int_{J'k} \frac{1}{1+t} \, dt
\]
\[
\gtrsim (N^{s-1}A^{-\frac{p+1}{4}})^{\alpha} \sum_{j_k \in \mathcal{P}} \frac{1}{1+t'_{jk}}.
\]
, where we used the elementary inequality \(\frac{1}{100R + |t-t'_{jk}|} \geq \frac{1}{100R + |t-t'_{jk}|} \geq \frac{1}{100R + |t'_{jk}|} \). Then it suffices to estimate \text{card} \mathcal{P}. Let \(j_{k_{\max}} := \max_{j_k \in \mathcal{P}} j_k\). By applying \text{card} \mathcal{P} - times \(5.32\), by the construction of \mathcal{P} and by \(5.26\) we have
\[
N^{2(1-s)A^{p+1}} \gtrsim E \left( I w(t'_{j_{k_{\max}}}), \bigcup_{j_k \in \mathcal{P}} B(x'_{jk}, R + |t'_{j_{k_{\max}}} - t'_{jk}|) \right)
\]
\[
\gtrsim \sum_{j_k \in \mathcal{P}} E \left( I w(t'_{jk}), B(x'_{jk}, R) \right) + o \left( (A^{-\frac{p+1}{4}} N^{s-1})^{\alpha} \right) \text{card} \mathcal{P}
\]
\[
\gtrsim \text{card} \mathcal{P} (A^{-\frac{p+1}{4}} N^{s-1})^{\alpha}.
\]
(5.3) follows.

5.3. **Proof of Lemma 5.3.** The proof is made of several steps:

- Long-time weighted decay estimates.

Let \(x_0 \in \mathbb{R}^3\).

We chop \(J'\) into subintervals \((J'_q)\) such that \(I w\) s.t. \(\|w\|_{L^2(\mathbb{R}^p) - L^2(\mathbb{R}^p)} = \eta_0, 1 \leq q < Q\) and \(\|w\|_{L^2(\mathbb{R}^p) - L^2(\mathbb{R}^p)} \leq \eta_0\). Then, by repeating the steps from (5.29) to (5.30), we get
\[
(5.34)
\int_{J'} \int_{\mathbb{R}^3} \frac{|w|^{p+1}}{|x-x_0|} \, dx dt \lesssim N^{2(1-s)A^{p+1}}
\]
Therefore (see Lemma 5.3, [15]) we have
We chop each subinterval

\[ \frac{1}{|t|} \int |w|^2 \frac{dx}{(x-x_0)^2} \] \[ \frac{n+1}{|t|} \int dt \leq X_1 + X_2 \]

with

\[ X_1 := \frac{1}{|t|} \int |w|^2 \frac{dx}{(x-x_0)^2} \frac{n+1}{|t|} \int dt \]

and

\[ X_2 := \frac{1}{|t|} \int |w|^2 \frac{dx}{(x-x_0)^2} \frac{n+1}{|t|} \int dt \]

But, by Hölder (in space) and (5.34) we have

\[ X_1 \lesssim \frac{1}{|t|} \int |w|^{p+1} \frac{dx}{(x-x_0)^2} \frac{n+1}{|t|} \int dt \]

and

\[ X_2 \lesssim \frac{1}{|t|} \int |w|^{p+1} \frac{dx}{(x-x_0)^2} \frac{n+1}{|t|} \int dt \lesssim (N^{2(1-s)} A^{p+1}) \frac{n+1}{|t|} \]

Therefore we have

\[ \int \frac{1}{|t|} \int |w|^2 \frac{dx}{(x-x_0)^2} \frac{n+1}{|t|} \int dt \lesssim (N^{2(1-s)} A^{p+1}) \frac{n+1}{|t|} \] \[ (5.35) \]

- We chop \( J \) into subintervals \( (J'_j = [t'_j, t'_j+1]) \) such that \( |Iw|_{L^2} \leq t'_j \) for all \( j \leq m \) such that \( |Iw|_{L^2} < t' \). Therefore, using (2.1), we get, by Proposition 2.1

\[ \hat{Z}(J'_j, w) \lesssim N^{1-s} A^{\frac{n+1}{2}} \] \[ (5.36) \]

By (5.2), there exists \( x'_j \in \mathbb{R}^3 \) and \( t'_j \in J'_j \) such that

\[ E(Iw(t'_j), B(x'_j, R)) \gtrsim (N^{s-1} A^{2n+1})^\alpha \] \[ (5.37) \]

and, by (5.33), we see that (with \( t'_a := t'_j \), \( t'_b := t \), \( t \in J'_j \)) there exists \( \eta_1 > 0 \) such that

\[ E(Iw(t), B(x'_j, R + |t-t'_j|)) \geq 3(N^{s-1} \eta_1 A^{2n+1})^\alpha \] \[ (5.38) \]

- We chop each subinterval \( J'_j \) into subintervals \( (J'_{j,k} = [t'_{j,k}, t'_{j,k+1}]) \) such that \( \hat{Z}(J'_{j,k}, w) \sim (N^{s-1} \eta_1 A^{2n+1})^\alpha \) and \( \hat{Z}(J'_{j,k}, w) \leq 3(N^{s-1} \eta_1 A^{2n+1})^\alpha \).

Notice that by (5.36),

\[ K_j \lesssim (N^{1-s} A^{2n+1})^\alpha \] \[ (5.39) \]

Next, given \( \epsilon < < 1 \), we claim that there exists \( C_0 > 0 \) such that

\[ \log(C_0) \lesssim (N^{2(1-s)} A^{p+1})^{\frac{n+1}{2}} \epsilon^{-(p+1)} \] \[ (5.40) \]
and such that for all $1 \leq k \leq K_j$ there exists $\tilde{t}_{j,k} \in [t_{j,k} - t_{j,k}^0] + C_0(t_{j,k} - t_{j,k}^0)$ such that (5.7) holds with $x := x_j$; if not, for all $C > 0$ such that $\log(C) >> e^{-\frac{1}{p+1}}(N^{2(1-s)}A^{p+1})^{\frac{1}{p+1}}$, we would have, by (5.35)

$$
(N^{2(1-s)}A^{p+1})^{\frac{1}{p+1}} \geq \int_{t_{j,k}}^{t_{j,k} + C(t_{j,k} - t_{j,k}^0)} \left\| \frac{f_w(t - t_{j,k}^0)}{(x - t_{j,k})} \right\|_{L^2(R^3)}^{p+1} dt \geq \log(C) e^{p+1}
$$

and it would yield a contradiction.

- Now we claim that there exists $(k_0, j_0)$ such that

$$
|t_{j_0, k_0 + 1} - t_{j_0, k_0}| \geq M' \left( R + |t_{j_0, k_0} - t_{j_0}^0| \right)
$$

with $M'$ such that

$$
(5.41) \quad M' - C_0 >> M + C_0
$$

If not, this would imply, by simple induction on $k$, that $|t_{j,k} - t_{j,k}^0| \leq (2M')^k R$ for all $j$ and therefore, by (5.39), we would have $|J_j^0| \leq (2M')^{C(N^{1-s}A^{\frac{p+1}{2}})^\alpha}$ for all $j$. But, by Lemma 5.2, this would imply that

$$
\log \left( \frac{1 + m(2M')^{C(N^{1-s}A^{\frac{p+1}{2}})^\alpha}}{(2M')^{C(N^{1-s}A^{\frac{p+1}{2}})^\alpha}} \right) \leq \sum_{j=1}^{m} \frac{1}{1 + (2M')^{C(N^{1-s}A^{\frac{p+1}{2}})^\alpha}} \leq (N^{1-s}A^{\frac{p+1}{2}})^\alpha.
$$

Therefore $\log(m) \leq C(2M')^{C(N^{1-s}A^{\frac{p+1}{2}})^\alpha}$ and, combining this inequality with (5.40) and (5.41), this yields a contradiction with (5.4). Consequently,

$$
\begin{align*}
t_{j_0, k_0 + 1} - t_{j_0, k_0} &\geq t_{j_0, k_0 + 1} - t_{j_0, k_0} - C_0(t_{j_0, k_0} - t_{j_0}^0) \\
&\geq M' \left( R + |t_{j_0, k_0} - t_{j_0}^0| \right) - C_0(t_{j_0, k_0} - t_{j_0}^0) \\
&\geq M \left( R + |t_{j_0, k_0} - t_{j_0}^0| \right)
\end{align*}
$$

and, choosing $R := R + |t_{j_0, k_0} - t_{j_0}^0|$, $t_l := t_{j_0, k_0}$ and $T := t_{j_0, k_0 + 1}$, we have (5.5), (5.6) and (5.7).

6. Proof of Proposition 2.3

In this section we prove Proposition 2.3. The proof is made of several steps:

- We divide $[T, b]$ into subintervals $(K_l = [t_l, t_{l+1}])_{1 \leq l \leq m}$ such that $\|w\|_{L^{2(p-1)}_w L^{2(p-1)}_x(K_l)} = \eta_0$ and $\|w\|_{L^{2(p-1)}_w L^{2(p-1)}_x(K_{l+1})} \leq \eta_0$, with $\eta_0$ defined in Proposition 2.1. Notice that, in view of (2.1), (2.3) and Proposition 2.1, we have $Z(w, K_l) \lesssim N^{1-s}A^{\frac{p+1}{2}}$ and, by iteration, in view of (2.3), we have

$$
(6.1) \quad Z(w, [T, b]) \lesssim L^2_w L^{2(p-1)}_x N^{1-s}A^{\frac{p+1}{2}}
$$
Again, we divide \([T, b]\) into subintervals \(J_j := [t_j, t_{j+1}]\) such that 
\[
\| w' \|_{L^2_0(L^p_{(p-1)})} \leq \eta_0, 1 \leq j \leq k \quad \text{and} \quad \| w' \|_{L^2_0(L^p_{(p-1)})} \leq \eta_0.
\]
Noticing that
\[
\begin{aligned}
\tilde{k} & \lesssim L^{2(p-1)}_w, \\
\text{in view of (2.4). Again, in view of (2.2), (2.4) and Proposition 2.1}
\end{aligned}
\]
\[
Z(w', [T, b]) \lesssim L^{2(p-1)}_w N^{-s} A^{1-w}.
\]

Let \(\Gamma := w - v - w'\). A simple computation shows that
\[
\begin{aligned}
\partial_t \Gamma - \triangle \Gamma &= (IF(w') - F(Iw')) + (F(Iw) - IF(w)) + (F(Iw') - F(Iw)) \\
\text{Therefore we conclude that} \quad \Gamma(t) &= \Gamma_1(t_j) + X_1 + X_2 + X_3,
\end{aligned}
\]
with
\[
\begin{aligned}
\Gamma_1(t_j, t) &:= \cos ((t-t_j)(D)) \Gamma(t_j) + \sin ((t-t_j)(D)) \partial_t \Gamma(t_j), \\
X_1 &:= -\int_t^{t_j} \sin ((t-s)(D)) (IF(w') - IF(w')) \, ds, \\
X_2 &:= -\int_t^{t_j} \sin ((t-s)(D)) (IF(w) - F(Iw')) \, ds, \\
X_3 &:= -\int_t^{t_j} \sin ((t-s)(D)) (F(Iw) - F(Iw')) \, ds.
\end{aligned}
\]
Notice that \(\Gamma_1(t_{j+1}, t) = \Gamma_1(t_j, t) + Y_1 + Y_2 + Y_3\), with
\[
\begin{aligned}
Y_1 &:= -\int_{t_j}^{t_{j+1}} \sin ((t-s)(D)) (IF(w') - IF(w')) \, ds, \\
Y_2 &:= -\int_{t_j}^{t_{j+1}} \sin ((t-s)(D)) (IF(w) - F(Iw')) \, ds, \\
Y_3 &:= -\int_{t_j}^{t_{j+1}} \sin ((t-s)(D)) (F(Iw) - F(Iw')) \, ds.
\end{aligned}
\]
In view of (2.19) and (2.4), it is enough to estimate 
\[
\begin{aligned}
\text{By the Sobolev embedding} \quad \| \Gamma \|_{L^2_0(L^p_{(p-1)})} &\lesssim \| (D)^{s_c - \frac{1}{2}} \Gamma \|_{L^2_0(L^p_{(p-1)})},
\end{aligned}
\]
and for all \(1 \leq j < k\), \(Z(t_j, t) := \sup_{(q, r)-\frac{1}{2}} - \text{wave adm} \| (D)^{s_c - \frac{1}{2}} \Gamma \|_{L^2_0(L^p_{(p-1)})} \| w' \|_{L^2_0(L^p_{(p-1)})}\), for all \(1 \leq j < k\), and for \(t_j \leq t \leq t_{j+1}\). Let \(Z(t_j) := \sup_{(q, r)-\frac{1}{2}} - \text{wave adm} \| (D)^{s_c - \frac{1}{2}} \Gamma_1(t_j) \|_{L^2_0(L^p_{(p-1)})}\),

\[
\begin{aligned}
Z(t_j, t) &\leq Z(t_j) + C \sup_{(q, r)-\frac{1}{2}} - \text{wave adm} \| (D)^{s_c - \frac{1}{2}} X_1 \|_{L^2_0(L^p_{(p-1)})} \| w' \|_{L^2_0(L^p_{(p-1)})} \\
&+ C \sup_{(q, r)-\frac{1}{2}} - \text{wave adm} \| (D)^{s_c - \frac{1}{2}} X_2 \|_{L^2_0(L^p_{(p-1)})} \| w' \|_{L^2_0(L^p_{(p-1)})} \\
&+ C \sup_{(q, r)-\frac{1}{2}} - \text{wave adm} \| (D)^{s_c - \frac{1}{2}} X_3 \|_{L^2_0(L^p_{(p-1)})} \| w' \|_{L^2_0(L^p_{(p-1)})},
\end{aligned}
\]
and
\[
\begin{aligned}
Z(t_{j+1}) &\leq Z(t_j) + C \sup_{(q, r)-\frac{1}{2}} - \text{wave adm} \| (D)^{s_c - \frac{1}{2}} Y_1 \|_{L^2_0(L^p_{(p-1)})} \\
&+ C \sup_{(q, r)-\frac{1}{2}} - \text{wave adm} \| (D)^{s_c - \frac{1}{2}} Y_2 \|_{L^2_0(L^p_{(p-1)})} \\
&+ C \sup_{(q, r)-\frac{1}{2}} - \text{wave adm} \| (D)^{s_c - \frac{1}{2}} Y_3 \|_{L^2_0(L^p_{(p-1)})}.
\end{aligned}
\]
We are interested in estimating \( \| (D)^{s_v} \frac{1}{2} X_2 \|_{L^2_t L^2_x([t_j, t])} \). We write \( X_2 := X_{2,1} + X_{2,2} \) with
\[
X_{2,1} := - \int_{t_j}^t \frac{\sin((t-s)D)}{(D)^{s_v} (F(u) - F(w))} \frac{Z_1}{(I - 1) (F(P_{< N} u))} ds
\]
\[
X_{2,2} := - \int_{t_j}^t \frac{\sin((t-s)D)}{(D)^{s_v} (F(u) - F(I w))} ds.
\]
First we deal with \( \| (D)^{s_v} \frac{1}{2} X_{2,2} \|_{L^2_t L^2_x([t_j, t])} \). By (6.1) we have
\[
\| (D)^{s_v} \frac{1}{2} X_{2,2} \|_{L^2_t L^2_x([t_j, t])} \lesssim \| (D)^{s_v} \frac{1}{2} (F(u) - F(I w)) \|_{L^2_t L^2_x([t_j, t])}
\]
\[
\lesssim \| u \|_{L^2_t L^2_x([t_j, t])} \| (D)^{s_v} \frac{1}{2} P_{< N} u \|_{L^4_t L^4_x([t_j, t])}
\]
\[
\lesssim \frac{\| (D)^{1-s} I u \|_{L^2_t L^2_x([t_j, t])} \| (D)^{s_v} \frac{1}{2} P_{< N} u \|_{L^4_t L^4_x([t_j, t])}}{N^{(1-s)\frac{1}{2}}}
\]
\[
\lesssim \left( \frac{L^2_t L^2_x}{} \right)^{\frac{1}{4}}.
\]
Next we deal with \( \| (D)^{s_v} \frac{1}{2} X_{2,1} \|_{L^2_t L^2_x([t_j, t])} \). It is enough to bound \( \| (D)^{s_v} \frac{1}{2} X_{2,1,1} \|_{L^2_t L^2_x([t_j, t])} \), \( \| (D)^{s_v} \frac{1}{2} X_{2,1,2} \|_{L^2_t L^2_x([t_j, t])} \) and \( \| (D)^{s_v} \frac{1}{2} X_{2,1,3} \|_{L^2_t L^2_x([t_j, t])} \) with
\[
X_{2,1,1} := - \int_{t_j}^t \frac{\sin((t-s)D)}{(D)^{s_v} (F(u) - F(P_{< N} w))} \frac{Z_1}{(I - 1) (F(P_{< N} w))} ds
\]
\[
X_{2,1,2} := - \int_{t_j}^t \frac{\sin((t-s)D)}{(D)^{s_v} (F(u) - F(P_{< N} w))} \frac{Z_2}{(I - 1) (\int_0^1 |P_{< N} w(s) + yP_{< N} w(s)|^{p-1} P_{< N} w(s) dy)} ds
\]
and
\[
X_{2,1,3} := - \int_{t_j}^t \frac{\sin((t-s)D)}{(D)^{s_v} (F(u) - F(P_{< N} w))} \frac{Z_2}{(I - 1) (\int_0^1 \frac{P_{< N} w + yP_{< N} w}{P_{< N} w + yP_{< N} w} |P_{< N} w + yP_{< N} w|^{p-1} P_{< N} w(s) dy)} ds.
\]
Again, we use the smoothness of \( F \) (is \( C^1 \)) to deal with \( X_{2,1,1} \). We have
\[
\| (D)^{s_v} \frac{1}{2} X_{2,1,1} \|_{L^2_t L^2_x([t_j, t])} \lesssim \| (D)^{s_v} \frac{1}{2} Z_1 \|_{L^2_t L^2_x([t_j, t])}
\]
\[
\lesssim \frac{\| \nabla F(P_{< N} w) \|_{L^2_t L^2_x([t_j, t])}}{N^{(1-s)\frac{1}{2}}}
\]
\[
\lesssim \frac{1}{N^{(1-s)\frac{1}{2}}} \| P_{< N} w \|_{L^2_t L^2_x([t_j, t])} \| \nabla P_{< N} w \|_{L^2_t L^2_x([t_j, t])}.
\]
As for \( X_{2,1,2} \), we have
In order to estimate \( \| (D)^{s_e - \frac{1}{2}} X_{2,1,2} \|_{L_t^2 L_x^\infty((t_j,t_l]} \leq \| (D)^{s_e - \frac{1}{2}} Z_e \|_{L_t^2 L_x^2((t_j,t_l]}} \)

\[
\lesssim \int_0^1 \left[ \| (D)^{s_e - \frac{1}{2}} \left( |P_{<N} w + y P_{\geq N} w|^p \right) \|_{L_x^4 L_{x,y}^{p-1}((T,b)})} + \| P_{<N} w + y P_{\geq N} w|_{L_x^4 L_{x,y}^{p-1}((T,b)})} \| (D)^{s_e - \frac{1}{2}} P_{\geq N} w \|_{L_t^4 L_x^4((T,b)])} \right] \, dy
\]

\[
\lesssim \frac{(L_x^{2(p-1)} - N^{1-s} A + \frac{1}{p})^p}{N^{(1-s)c_2}}
\]

by using the product rule followed by a two-variable Leibnitz rule

(see Appendix A with \( f := P_{<N} w \) and \( g := P_{\geq N} w \), \( F_g(f,g) = |f + yg|^p \)

and \( \lambda = p - 2 \).

\( X_{2,1,3} \) is treated in a similar fashion: we get

\[
\| (D)^{s_e - \frac{1}{2}} X_{2,1,3} \|_{L_t^1 L_x^\infty((t_j,t_l]} \leq \frac{(L_x^{2(p-1)} - N^{1-s} A + \frac{1}{p})^p}{N^{(1-s)c_2}}.
\]

Finally we have

\[
\| (D)^{s_e - \frac{1}{2}} X_{2} \|_{L_t^1 L_x^\infty((t_j,t_l]} \leq \frac{(L_x^{2(p-1)} - N^{1-s} A + \frac{1}{p})^p}{N^{(1-s)c_2}}.
\]

Similarly

\[
\| (D)^{s_e - \frac{1}{2}} Y_{2} \|_{L_t^1 L_x^\infty((t_j,t_l]} \leq \frac{(L_x^{2(p-1)} - N^{1-s} A + \frac{1}{p})^p}{N^{(1-s)c_2}}.
\]

\( \bullet \) In order to estimate \( \| (D)^{s_e - \frac{1}{2}} X_{1} \|_{L_t^1 L_x^\infty((t_j,t_l]} \) and \( \| (D)^{s_e - \frac{1}{2}} Y_{1} \|_{L_t^1 L_x^\infty((t_j,t_l]} \), we perform a similar decomposition to that in the proof of (6.7), using (6.3) instead of (6.1).

We have

\[
\| (D)^{s_e - \frac{1}{2}} X_{1} \|_{L_t^1 L_x^\infty((t_j,t_l]} \leq \frac{(L_x^{2(p-1)} - N^{1-s} A + \frac{1}{p})^p}{N^{(1-s)c_2}}.
\]

and

\[
\| (D)^{s_e - \frac{1}{2}} Y_{1} \|_{L_t^1 L_x^\infty((t_j,t_l]} \leq \frac{(L_x^{2(p-1)} - N^{1-s} A + \frac{1}{p})^p}{N^{(1-s)c_2}}.
\]

\( \bullet \) We are interested in estimating \( \| (D)^{s_e - \frac{1}{2}} X_3 \|_{L_t^1 L_x^\infty((t_j,t_l]} \) and \( \| (D)^{s_e - \frac{1}{2}} Y_3 \|_{L_t^1 L_x^\infty((t_j,t_l]} \).

Since \( s_e > \frac{1}{2} \), there exist \( \theta := \theta(p) > 0 \), \( m < \frac{1}{2} \), \( (q,r) \) \( m \)-wave admissible such that

\[
\| (D)^{s_e - \frac{1}{2}} I v \|_{L_t^q L_x^r((t_j,t_l]} \leq \| I v \|_{L_t^q L_x^r((t_j,t_l]}^{\theta} \| (D)^{1-m} I v \|_{L_t^q L_x^r}^{1-\theta}
\]

in view of (2.19) and (2.21). Therefore
In this section we prove a two-variable Leibnitz rule 

\[ \frac{d}{dt} \left( \frac{d}{dz} \right) \left( \frac{d}{dz} \right) = \frac{d}{dz} \left( \frac{d}{dt} \right) + \frac{d}{dt} \left( \frac{d}{dz} \right) + \frac{d}{dz} \left( \frac{d}{dt} \right) \]

provided that

\[ 2C^2 \left( \frac{d}{dz} \right) \left( \frac{d}{dt} \right) \left( \frac{d}{dz} \right) \left( \frac{d}{dt} \right) \]

for some \( \lambda > 0 \). Then

\[ \| F(f, g) \|_{H^{s,p}} \lesssim \| f \|_{L^n} \| f \|_{H^{s,p}} + \| g \|_{L^n} \| f \|_{H^{s,p}} \]

for all \( (f, g) \in C^4 \).
assuming that \((p,p_1,p_2,\varphi_1,\varphi_2,r_1,r_2,\tilde{r}_1,\tilde{r}_2) \in (1, \infty)^9\) and
\[
\frac{1}{p} = \frac{\lambda}{p_1} + \frac{1}{p_2} = \frac{\lambda}{r_1} + \frac{1}{r_2}
\]

Proof. The proof relies upon a simple modification of the one-variable fractional Leibnitz rules (see [6]). We recall the following inequalities (see for example [21]):

\[
\begin{aligned}
\left( \frac{N_1}{N_1^*} \right)^{\frac{1}{\lambda}} M_h(\tilde{P}_{N_1} q)(x) \\
\left( \frac{N_2}{N_2^*} \right)^{\frac{1}{\lambda}} M_h(\tilde{P}_{N_2} q)(x) \\
\left( \frac{N_3}{N_3^*} \right)^{\frac{1}{\lambda}} M_h(\tilde{P}_{N_3} q)(x)
\end{aligned}
\]

with \((N_1, N_2) \in 2^{2\pi} \subset \mathbb{N}, H a nonnegative function, \psi_M(\xi) := \psi \left( \frac{\xi}{M} \right) \text{ (if } M \in 2^{2\pi})\),

\[
\tilde{P}_M := P_{M \leq 2M} \text{ (if } M \in 2^{2\pi}), \tilde{P}_1 := P_{\leq 2} \text{ and } (M_h(f))(x) := \sup_{r > 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy.
\]

Recall also the Paley-Littlewood inequalities (see [17])

\[
\begin{aligned}
\|F(g)\|_{H^{p}} \lesssim \|P_{1}(F(f,g))\|_{L^{p}} + \left\| \left( \sum_{N_1 \in 2^{\pi^*}} N^{2^*}_{1} \right) |P_{N_1}(F(f,g))|^2 \right\|_{L^{p}}. \\
\end{aligned}
\]

and

\[
\begin{aligned}
\left\| \left( \sum_{N_1 \in 2^{\pi^*}} N^{2^*}_{1} |P_{N_1} f|^2 \right) \right\|_{L^{p}} \lesssim \|f\|_{H^{p^*}}.
\end{aligned}
\]

We write

\[
\begin{aligned}
P_{N_1}(F(f,g))(x) &= \int_{\mathbb{R}^3} F(f(y),g(y)) \psi_{N_1}(x-y) dy \\
&= \int_{\mathbb{R}^3} (F(f(y),g(y)) - F(f(x),g(x))) \psi_{N_1}(x-y) dy \\
&= A_1 + A_2 + A_3 + A_4
\end{aligned}
\]

with

\[
\begin{aligned}
A_1 := \int_{0}^{1} \int_{\mathbb{R}^3} \partial_x F(\mu f(y) + (1 - \mu)f(x), \mu g(y) + (1 - \mu)g(x)) |f(y) - f(x)| |\psi_{N_1}(x-y)| d\mu dy \\
A_2 := \int_{0}^{1} \int_{\mathbb{R}^3} \partial_y F(\mu f(y) + (1 - \mu)f(x), \mu g(y) + (1 - \mu)g(x)) |f(y) - f(x)| |\psi_{N_1}(x-y)| d\mu dy \\
A_3 := \int_{0}^{1} \int_{\mathbb{R}^3} \partial_y F(\mu f(y) + (1 - \mu)f(x), \mu g(y) + (1 - \mu)g(x)) |g(y) - g(x)| |\psi_{N_1}(x-y)| d\mu dy \\
A_4 := \int_{0}^{1} \int_{\mathbb{R}^3} \partial_y F(\mu f(y) + (1 - \mu)f(x), \mu g(y) + (1 - \mu)g(x)) |g(y) - g(x)| |\psi_{N_1}(x-y)| d\mu dy
\end{aligned}
\]

Let us deal for example with \(A_1\).

\[
\sum_{N_1 \in 2^{\pi^*}} N^{2^*}_{1} A^{2}_{1} \lesssim \sum_{N_1 \in 2^{\pi^*}} N^{2^*}_{1} \left( \int_{\mathbb{R}^3} |f(y)|^2 |f(y) - f(x)| |\psi_{N_1}(x-y)| dy \right)^2
\]

\[
+ \sum_{N_1 \in 2^{\pi^*}} N^{2^*}_{1} \left( \int_{\mathbb{R}^3} |f(y)|^2 |f(y) - f(x)| |\psi_{N_1}(x-y)| dy \right)^2
\]

\[
+ \sum_{N_1 \in 2^{\pi^*}} N^{2^*}_{1} \left( \int_{\mathbb{R}^3} |g(y)|^2 |f(y) - f(x)| |\psi_{N_1}(x-y)| dy \right)^2
\]

\[
+ \sum_{N_1 \in 2^{\pi^*}} N^{2^*}_{1} \left( \int_{\mathbb{R}^3} |g(y)|^2 |f(y) - f(x)| |\psi_{N_1}(x-y)| dy \right)^2
\]

\[
\lesssim A^{2}_{1,1} + A^{2}_{1,2} + A^{2}_{1,3} + A^{2}_{1,4}
\]

We have
and therefore (7.7) also holds if $A > s_1$ (7.4) we have and therefore, by Fefferman-Stein maximal inequality \cite{8}, Hölder’s inequality and which implies that

$$\sum_{N_1 \in 2^{s_1^*}} N_{1}^{2s} \left( \sum_{N_1 \leq N_1} \frac{N_{1}}{N_{1}^2} \left( \mathcal{H}_N(f)(x) \right)^2 \right) \leq A_{1,1}^2 + A_{1,2}^2$$

But, by (7.3)

$$A_{1,1}^2 \lesssim \left( \mathcal{K}_N(f)(x) \right)^2 \sum_{N_1 \in 2^{s_1^*}} N_{1}^{2s} \left( \sum_{N_1 \leq N_1} \frac{N_{1}}{N_{1}^2} \left( \mathcal{H}_N(f)(x) \right)^2 \right)$$

Now, by Young’s inequality we have (since $s < 1$)

$$\sum_{N_1 \in 2^{s_1^*}} N_{1}^{2s} \left( \sum_{N_1 \leq N_1} \frac{N_{1}}{N_{1}^2} \right)^2 \leq \sum_{N_1 \in 2^{s_1^*}} N_{1}^{2s} \left( \sum_{N_1 \leq N_1} \frac{N_{1}}{N_{1}^2} \right)^2 \leq \sum_{N_1 \in 2^{s_1^*}} N_{1}^{2s} \left| a_{N_1} \right|^2$$

which implies that

$$A_{1,1}^2 \lesssim \left( \mathcal{K}_N(f)(x) \right)^2 \sum_{N_1 \in 2^{s_1^*}} N_{1}^{2s} \left( \sum_{N_1 \leq N_1} \frac{N_{1}}{N_{1}^2} \left( \mathcal{H}_N(f)(x) \right)^2 \right)$$

and therefore, by Fefferman-Stein maximal inequality \cite{8}, Hölder’s inequality and (7.4) we have

$$\| A_{1,1} \|_{L^p} \lesssim \| f \|_{L^{p_1}} \| f \|_{H^{s,p_2}}$$

Also, by (7.3) we have

$$A_{1,2}^2 \lesssim \left( \mathcal{K}_N(f)(x) \right)^2 \sum_{N_1 \in 2^{s_1^*}} N_{1}^{2s} \left( \sum_{N_1 \leq N_1} \frac{N_{1}}{N_{1}^2} \left( \mathcal{H}_N(f)(x) \right)^2 \right)$$

But, by Young’s inequality (since $s > 0$)

$$\sum_{N_1 \in 2^{s_1^*}} N_{1}^{2s} \left( \sum_{N_1 \geq N_1} \left| a_{N_1} \right|^2 \right)^2 \leq \sum_{N_1 \in 2^{s_1^*}} N_{1}^{2s} \left( \sum_{N_1 \geq N_1} \left( \frac{N_{1}}{N_{2}} \right)^2 \sum_{N_2} \left| a_{N_2} \right|^2 \right)^2$$

and therefore (7.7) also holds if $A_{1,1}^2$ is substituted for $A_{1,2}^2$.

The other terms ($A_{1,2}$, $A_{1,3}$, $A_{1,4}$ and then $A_2$, $A_3$, $A_4$) are treated in a similar fashion.

We also have $\| P_1(F(f,g)) \|_{L^p} \lesssim \| F(f,g) \|_{L^p}$. Then writing $F(f,g) = F(f,g) - F(0,0)$ and applying the fundamental theorem of calculus, we see that (7.2) holds if $s = 0$. 

\[ \square \]
REFERENCES


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